We now take a break from the statistical process. Why? In Chapter 1, we mentioned that inferential statistics uses methods that generalize results obtained from a sample to the population and measures their reliability. But how can we measure their reliability? It turns out that the methods we use to generalize results from a sample to a population are based on probability and probability models. Probability is a measure of the likelihood that something occurs. This part of the course will focus on methods for determining probabilities.
5 Probability

Outline
5.1 Probability Rules
5.2 The Addition Rule and Complements
5.3 Independence and the Multiplication Rule
5.4 Conditional Probability and the General Multiplication Rule
5.5 Counting Techniques
5.6 Putting It Together: Which Method Do I Use?
5.7 Bayes’s Rule (on CD)

MAKING AN INFORMED DECISION

Have you ever watched a sporting event on television in which the announcer cites an obscure statistic? Where do these numbers come from? Well, pretend that you are the statistician for your favorite sports team. Your job is to compile strange or obscure probabilities regarding your favorite team and a competing team. See the Decisions project on page 326.

PUTTING IT TOGETHER

In Chapter 1, we learned the methods of collecting data. In Chapters 2 through 4, we learned how to summarize raw data using tables, graphs, and numbers. As far as the statistical process goes, we have discussed the collecting, organizing, and summarizing parts of the process.

Before we can proceed with the analysis of data, we introduce probability, which forms the basis of inferential statistics. Why? Well, we can think of the probability of an outcome as the likelihood of observing that outcome. If something has a high likelihood of happening, it has a high probability (close to 1). If something has a small chance of happening, it has a low probability (close to 0). For example, in rolling a single die, it is unlikely that we would roll five straight sixes, so this result has a low probability. In fact, the probability of rolling five straight sixes is 0.00001286. So, if we were playing a game that entailed throwing a single die, and one of the players threw five sixes in a row, we would consider the player to be lucky (or a cheater) because it is such an unusual occurrence. Statisticians use probability in the same way. If something occurs that has a low probability, we investigate to find out “what’s up.”
5.1 Probability Rules

Preparing for This Section  Before getting started, review the following:
- Relative frequency (Section 2.1, p. 68)

**Objectives**
- Apply the rules of probabilities
- Compute and interpret probabilities using the empirical method
- Compute and interpret probabilities using the classical method
- Use simulation to obtain data based on probabilities
- Recognize and interpret subjective probabilities

**Note to Instructor**
If you like, you can print out and distribute the Preparing for This Section quiz located in the Instructor’s Resource Center. The purpose of the quiz is to verify that the students have the prerequisite knowledge for the section.

**Note to Instructor**
Probabilities can be expressed as fractions, decimals, or percents. You may want to give a brief review of how to convert from one form to the other.

**Figure 1**

![Graphs showing the proportion of heads observed after each flip of a coin](image)

**In Other Words**
Probability describes how likely it is that some event will happen. If we look at the proportion of times an event has occurred over a long period of time (or over a large number of trials), we can be more certain of the likelihood of its occurrence.

Looking at the graphs in Figures 1(a) and (b), we notice that in the short term (fewer flips of the coin) the observed proportion of heads is different and unpredictable for each experiment. As the number of flips of the coin increases, however, both graphs tend toward a proportion of 0.5. This is the basic premise of probability. Probability deals with experiments that yield random short-term results or outcomes yet reveal long-term predictability. **The long-term proportion with which a certain outcome is observed is the probability of that outcome.** So we say that the probability of observing a head is \( \frac{1}{2} \) or 50% or 0.5 because, as we flip the coin more times, the proportion of heads tends toward \( \frac{1}{2} \). This phenomenon is referred to as the **Law of Large Numbers.**

**The Law of Large Numbers**
As the number of repetitions of a probability experiment increases, the proportion with which a certain outcome is observed gets closer to the probability of the outcome.
The Law of Large Numbers is illustrated in Figure 1. For a few flips of the coin, the proportion of heads fluctuates wildly around 0.5, but as the number of flips increases, the proportion of heads settles down near 0.5. Jakob Bernoulli (a major contributor to the field of probability) believed that the Law of Large Numbers was common sense. This is evident in the following quote from his text 

*Ars Conjectandi*:

“For even the most stupid of men, by some instinct of nature, by himself and without any instruction, is convinced that the more observations have been made, the less danger there is of wandering from one’s goal.”

In probability, an **experiment** is any process with uncertain results that can be repeated. The result of any single trial of the experiment is not known ahead of time. However, the results of the experiment over many trials produce regular patterns that enable us to predict with remarkable accuracy. For example, an insurance company cannot know ahead of time whether a particular 16-year-old driver will be involved in an accident over the course of a year. However, based on historical records, the company can be fairly certain that about three out of every ten 16-year-old male drivers will be involved in a traffic accident during the course of a year. Therefore, of the 816,000 male 16-year-old drivers (816,000 repetitions of the experiment), the insurance company is fairly confident that about 30%, or 244,800, of the drivers will be involved in an accident. This prediction forms the basis for establishing insurance rates for any particular 16-year-old male driver.

We now introduce some terminology that we will need to study probability.

### Definitions

**In Other Words**

An outcome is the result of one trial of a probability experiment. The sample space is a list of all possible results of a probability experiment.

**EXAMPLE 1**

**Identifying Events and the Sample Space of a Probability Experiment**

**Problem:** A probability experiment consists of rolling a single *fair* die.

(a) Identify the outcomes of the probability experiment.

(b) Determine the sample space.

(c) Define the event $E = “\text{roll an even number}.”$

**Approach:** The outcomes are the possible results of the experiment. The sample space is a list of all possible outcomes.

**Solution**

(a) The outcomes from rolling a single fair die are $e_1 = “\text{rolling a one}” = \{1\}$, $e_2 = “\text{rolling a two}” = \{2\}$, $e_3 = “\text{rolling a three}” = \{3\}$, $e_4 = “\text{rolling a four}” = \{4\}$, $e_5 = “\text{rolling a five}” = \{5\}$, and $e_6 = “\text{rolling a six}” = \{6\}$.

(b) The set of all possible outcomes forms the sample space, $S = \{1, 2, 3, 4, 5, 6\}$. There are 6 outcomes in the sample space.

(c) The event $E = “\text{roll an even number}” = \{2, 4, 6\}$.

### Apply the Rules of Probabilities

Probabilities have some rules that must be satisfied. In these rules, the notation $P(E)$ means “the probability that event $E$ occurs.”
**Section 5.1 Probability Rules**

**In Other Words**
Rule 1 states that probabilities less than 0 or greater than 1 are not possible. Therefore, probabilities such as 1.32 or -0.5 are not possible. Rule 2 states when the probabilities of all outcomes are added, the sum must be 1.

**Rules of Probabilities**

1. The probability of any event $E$, $P(E)$, must be greater than or equal to 0 and less than or equal to 1. That is, $0 \leq P(E) \leq 1$.
2. The sum of the probabilities of all outcomes must equal 1. That is, if the sample space $S = \{e_1, e_2, \ldots, e_n\}$, then
   $$P(e_1) + P(e_2) + \cdots + P(e_n) = 1$$

A **probability model** lists the possible outcomes of a probability experiment and each outcome’s probability. A probability model must satisfy rules 1 and 2 of the rules of probabilities.

**EXAMPLE 2**

**A Probability Model**

In a bag of plain M&M milk chocolate candies, the colors of the candies can be brown, yellow, red, blue, orange, or green. Suppose that a candy is randomly selected from a bag. Table 1 shows each color and the probability of drawing that color.

To verify that this is a probability model, we must show that rules 1 and 2 of the rules of probabilities are satisfied.

Each probability is greater than or equal to 0 and less than or equal to 1, so rule 1 is satisfied.

Because

$$0.13 + 0.14 + 0.13 + 0.24 + 0.20 + 0.16 = 1$$

rule 2 is also satisfied. The table is an example of a probability model.

If an event is **impossible**, the probability of the event is 0. If an event is a **certainty**, the probability of the event is 1. The closer a probability is to 1, the more likely the event will occur. The closer a probability is to 0, the less likely the event will occur. For example, an event with probability 0.8 is more likely to occur than an event with probability 0.75. An event with probability 0.8 will occur about 80 times out of 100 repetitions of the experiment, whereas an event with probability 0.75 will occur about 75 times out of 100.

Be careful of this interpretation. Just because an event has a probability of 0.75 does not mean that the event must occur 75 times out of 100. It means that we expect the number of occurrences to be close to 75 in 100 trials of the experiment. The more repetitions of the probability experiment, the closer the proportion with which the event occurs will be to 0.75 (the Law of Large Numbers).

One goal of this course is to learn how probabilities can be used to identify **unusual events**.

**Definition**

An **unusual event** is an event that has a low probability of occurring.

Typically, an event with a probability less than 0.05 (or 5%) is considered unusual, but this **cutoff point** is not set in stone. The researcher and the context of the problem determine the probability that separates unusual events from **not so unusual events**.

For example, suppose that the probability of being wrongly convicted of a capital crime punishable by death is 3%. Even though 3% is below our 5% cutoff point, this probability is too high in light of the consequences (death for the wrongly convicted), so the event is not unusual (unlikely) enough. We would want this probability to be much closer to zero.

Now suppose that you are planning a picnic on a day for which there is a 3% chance of rain. In this context, you would consider “rain” an unusual (unlikely) event and proceed with the picnic plans.

---

**Table 1**

<table>
<thead>
<tr>
<th>Color</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>Brown</td>
<td>0.13</td>
</tr>
<tr>
<td>Yellow</td>
<td>0.14</td>
</tr>
<tr>
<td>Red</td>
<td>0.13</td>
</tr>
<tr>
<td>Blue</td>
<td>0.24</td>
</tr>
<tr>
<td>Orange</td>
<td>0.20</td>
</tr>
<tr>
<td>Green</td>
<td>0.16</td>
</tr>
</tbody>
</table>

*Source: M&Ms*

**Note to Instructor**

The interpretation of probability given here is important to emphasize. We will use this interpretation again when we discuss confidence intervals.
The point is this: Selecting a probability that separates unusual events from not so unusual events is subjective and depends on the situation. Statisticians typically use cutoff points of 0.01, 0.05, and 0.10. For many circumstances, any event that occurs with a probability of 0.05 or less will be considered unusual.

Next, we introduce three methods for determining the probability of an event: (1) the empirical method, (2) the classical method, and (3) the subjective method.

### Compute and Interpret Probabilities Using the Empirical Method

Because probabilities deal with the long-term proportion with which a particular outcome is observed, it makes sense that we begin our discussion of determining probabilities using the idea of relative frequency. Probabilities computed in this manner rely on empirical evidence, that is, evidence based on the outcomes of a probability experiment.

#### Approximating Probabilities Using the Empirical Approach

The probability of an event \( E \) is approximately the number of times event \( E \) is observed divided by the number of repetitions of the experiment.

\[
P(E) \approx \frac{\text{relative frequency of } E}{\text{number of trials of experiment}}
\]

The probability obtained using the empirical approach is approximate because different runs of the probability experiment lead to different outcomes and, therefore, different estimates of \( P(E) \). Consider flipping a coin 20 times and recording the number of heads. Use the results of the experiment to estimate the probability of obtaining a head. Now repeat the experiment. Because the results of the second run of the experiment do not necessarily yield the same results, we cannot say the probability equals the relative frequency; rather we say the probability is approximately the relative frequency. As we increase the number of trials of a probability experiment, our estimate becomes more accurate (again, the Law of Large Numbers).

#### EXAMPLE 3

**Using Relative Frequencies to Approximate Probabilities**

A pit boss wanted to approximate the probability of rolling a seven using a pair of dice that have been in use for a while. To do this he rolls the dice 100 times and records 15 sevens. The probability of rolling a seven is approximately \( \frac{15}{100} = 0.15 \).

When we survey a random sample of individuals, the probabilities computed from the survey are approximate. In fact, we can think of a survey as a probability experiment, since the results of a survey are likely to be different each time the survey is conducted because different people are included.

#### EXAMPLE 4

**Building a Probability Model from Survey Data**

**Problem:** The data in Table 2 represent the results of a survey in which 200 people were asked their means of travel to work.

(a) Use the survey data to build a probability model for means of travel to work.

(b) Estimate the probability that a randomly selected individual carpool to work. Interpret this result.

(c) Would it be unusual to randomly select an individual who walks to work?

**Approach:** To build a probability model, we estimate the probability of each outcome by determining its relative frequency.

<table>
<thead>
<tr>
<th>Means of Travel</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>Drive alone</td>
<td>153</td>
</tr>
<tr>
<td>Carpool</td>
<td>22</td>
</tr>
<tr>
<td>Public transportation</td>
<td>10</td>
</tr>
<tr>
<td>Walk</td>
<td>5</td>
</tr>
<tr>
<td>Other means</td>
<td>3</td>
</tr>
<tr>
<td>Work at home</td>
<td>7</td>
</tr>
</tbody>
</table>
Section 5.1 Probability Rules

Solution
(a) There are 153 + 22 + \cdots + 7 = 200 individuals in the survey. The individuals can be thought of as trials of the probability experiment. The relative frequency for “drive alone” is \(\frac{153}{200} = 0.765\). We compute the relative frequency of the other outcomes similarly and obtain the probability model in Table 3.

(b) From Table 3, we estimate the probability to be 0.11 that a randomly selected individual carpools to work.

(c) The probability that an individual walks to work is approximately 0.025. It is somewhat unusual to randomly choose a person who walks to work.

Table 3
<table>
<thead>
<tr>
<th>Means of Travel</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>Drive alone</td>
<td>0.765</td>
</tr>
<tr>
<td>Carpool</td>
<td>0.11</td>
</tr>
<tr>
<td>Public transportation</td>
<td>0.05</td>
</tr>
<tr>
<td>Walk</td>
<td>0.025</td>
</tr>
<tr>
<td>Other means</td>
<td>0.015</td>
</tr>
<tr>
<td>Work at home</td>
<td>0.035</td>
</tr>
</tbody>
</table>

Compute and Interpret Probabilities Using the Classical Method

When using the empirical method, we obtain an approximate probability of an event by conducting a probability experiment.

The classical method of computing probabilities does not require that a probability experiment actually be performed. Rather, we use counting techniques to determine the probability of an event.

The classical method of computing probabilities requires equally likely outcomes. An experiment is said to have equally likely outcomes when each outcome has the same probability of occurring. For example, in throwing a fair die once, each of the six outcomes in the sample space, \(\{1, 2, 3, 4, 5, 6\}\), has an equal chance of occurring. Contrast this situation with a loaded die in which a five or six is twice as likely to occur as a one, two, three, or four.

Computing Probability Using the Classical Method

If an experiment has \(n\) equally likely outcomes and if the number of ways that an event \(E\) can occur is \(m\), then the probability of \(E\), \(P(E)\), is

\[
P(E) = \frac{\text{number of ways that } E \text{ can occur}}{\text{number of possible outcomes}} = \frac{m}{n} \quad (2)
\]

So, if \(S\) is the sample space of this experiment,

\[
P(E) = \frac{N(E)}{N(S)} \quad (3)
\]

where \(N(E)\) is the number of outcomes in \(E\), and \(N(S)\) is the number of outcomes in the sample space.

EXAMPLE 5

Computing Probabilities Using the Classical Method

Problem: A pair of fair dice is rolled.
(a) Compute the probability of rolling a seven.
(b) Compute the probability of rolling “snake eyes”; that is, compute the probability of rolling a two.
(c) Comment on the likelihood of rolling a seven versus rolling a two.

Approach: To compute probabilities using the classical method, we count the number of outcomes in the sample space and count the number of ways the event can occur.

Solution
(a) In rolling a pair of fair dice, there are 36 equally likely outcomes in the sample space, as shown in Figure 2.
Chapter 5 Probability

So, the event has six outcomes, so using Formula (3), the probability of rolling a seven is

\[
P(E) = P(\text{roll a seven}) = \frac{N(E)}{N(S)} = \frac{6}{36} = \frac{1}{6}
\]

(b) The event \( F = \{\text{roll a two}\} \) has one outcome, so \( N(F) = 1 \). Using Formula (3), the probability of rolling a two is

\[
P(F) = P(\text{roll a two}) = \frac{N(F)}{N(S)} = \frac{1}{36}
\]

c) Since \( P(\text{roll a seven}) = \frac{6}{36} \) and \( P(\text{roll a two}) = \frac{1}{36} \), rolling a seven is six times as likely as rolling a two. In other words, in 36 rolls of the dice, we expect to observe about 6 sevens and only 1 two.

If we compare the empirical probability of rolling a seven, 0.15, obtained in Example 3, to the classical probability of rolling a seven, \( \frac{1}{6} \approx 0.167 \), obtained in Example 5(a), we see that they are not too far apart. In fact, if the dice are fair, we expect the relative frequency of sevens to get closer to 0.167 as we increase the number of rolls of the dice. That is, if the dice are fair, the empirical probability will get closer to the classical probability as the number of trials of the experiment increases. If the two probabilities do not get closer together, we may suspect that the dice are not fair.

In simple random sampling, each individual has the same chance of being selected. Therefore, we can use the classical method to compute the probability of obtaining a specific sample.

**Example 6**

**Computing Probabilities Using Equally Likely Outcomes**

**Problem:** Sophia has three tickets to a concert. Yolanda, Michael, Kevin, and Marissa have all stated they would like to go to the concert with Sophia. To be fair, Sophia decides to randomly select the two people who can go to the concert with her.

(a) Determine the sample space of the experiment. In other words, list all possible simple random samples of size \( n = 2 \).

(b) Compute the probability of the event “Michael and Kevin attend the concert.”
(c) Compute the probability of the event “Marissa attends the concert.”
(d) Interpret the probability in part (c).

**Approach:** First, we determine the outcomes in the sample space by making a table. The probability of an event is the number of outcomes in the event divided by the number of outcomes in the sample space.

**Solution**

(a) The sample space is listed in Table 4.

<table>
<thead>
<tr>
<th>Yolanda, Michael</th>
<th>Yolanda, Kevin</th>
<th>Yolanda, Marissa</th>
</tr>
</thead>
<tbody>
<tr>
<td>Michael, Kevin</td>
<td>Michael, Marissa</td>
<td>Kevin, Marissa</td>
</tr>
</tbody>
</table>

(b) We have $N(S) = 6$, and there is one way the event “Michael and Kevin attend the concert” can occur. Therefore, the probability that Michael and Kevin attend the concert is $\frac{1}{6}$.

(c) We have $N(S) = 6$, and there are three ways the event “Marissa attends the concert” can occur. The probability that Marissa will attend is $\frac{3}{6} = 0.5 = 50%$.

(d) If we conducted this experiment many times, about 50% of the experiments would result in Marissa attending the concert.

**Comparing the Classical Method and Empirical Method**

**Problem:** Suppose that a survey is conducted in which 500 families with three children are asked to disclose the gender of their children. Based on the results, it was found that 180 of the families had two boys and one girl.

(a) Estimate the probability of having two boys and one girl in a three-child family using the empirical method.

(b) Compute and interpret the probability of having two boys and one girl in a three-child family using the classical method, assuming boys and girls are equally likely.

**Approach:** To answer part (a), we determine the relative frequency of the event “two boys and one girl.” To answer part (b), we must count the number of ways the event “two boys and one girl” can occur and divide this by the number of possible outcomes for this experiment.

**Solution**

(a) The empirical probability of the event $E = \text{“two boys and one girl”}$ is

$$P(E) \approx \text{relative frequency of } E = \frac{180}{500} = 0.36 = 36\%$$

There is about a 36% probability that a family of three children will have two boys and one girl.

(b) To determine the sample space, we construct a tree diagram to list the equally likely outcomes of the experiment. We draw two branches corresponding to the two possible outcomes (boy or girl) for the first repetition of the experiment (the first child). For the second child, we draw four branches: two branches originate from the first boy and two branches originate from the first girl. This is repeated for the third child. See Figure 3, where B stands for boy and G stands for girl.
Chapter 5 Probability

The sample space $S$ of this experiment is found by following each branch to identify all the possible outcomes of the experiment:

So, $E = \text{"two boys and a girl"}$ = {BBG, BGB, GBB}, we have $N(E) = 3$. Since the outcomes are equally likely (for example, BBG is just as likely as BGB), the probability of $E$ is

$$P(E) = \frac{N(E)}{N(S)} = \frac{3}{8} = 0.375 = 37.5\%$$

There is a 37.5% probability that a family of three children will have two boys and one girl. If we repeated this experiment 1000 times and the outcomes are equally likely (having a girl is just as likely as having a boy), we would expect about 375 of the trials to result in 2 boys and 1 girl.

In comparing the results of Examples 7(a) and 7(b), we notice that the two probabilities are slightly different. Empirical probabilities and classical probabilities often differ in value. As the number of repetitions of a probability experiment increases, the empirical probability should get closer to the classical probability. That is, the classical probability is the theoretical relative frequency of an event after a large number of trials of the probability experiment. However, it is also possible that the two probabilities differ because having a boy or having a girl are not equally likely events. (Maybe the probability of having a boy is 50.5% and the probability of having a girl is 49.5%.) If this is the case, the empirical probability will not get closer to the classical probability.

### Use Simulation to Obtain Data Based on Probabilities

Suppose that we want to determine the probability of having a boy. Using classical methods, we would assume that having a boy is just as likely as having a girl, so the probability of having a boy is 0.5. We could also approximate this probability by
looking in the *Statistical Abstract of the United States* under Vital Statistics and determining the number of boys and girls born for the most recent year for which data are available. In 2005, for example, 2,119,000 boys and 2,020,000 girls were born. Based on empirical evidence, the probability of a boy is approximately

\[
\frac{2,119,000}{2,119,000 + 2,020,000} = 0.512 \approx 51.2\%.
\]

Notice that the empirical evidence, which is based on a very large number of repetitions, differs from the value of 0.50 used for classical methods (which assumes boys and girls are equally likely). This empirical results serve as evidence against the belief that the probability of having a boy is 0.5.

Instead of obtaining data from existing sources, we could also simulate a probability experiment using a graphing calculator or statistical software to replicate the experiment as many times as we like. Simulation is particularly helpful for estimating the probability of more complicated events. In Example 7 we used a tree diagram and the classical approach to find the probability of an event (a three-child family will have two boys and one girl). In this next example, we use simulation to estimate the same probability.

### Example 8

**Simulating Probabilities**

**Problem**

(a) Simulate the experiment of sampling 100 three-child families to estimate the probability that a three-child family has two boys.

(b) Simulate the experiment of sampling 1,000 three-child families to estimate the probability that a three-child family has two boys.

**Approach:** To simulate probabilities, we use a random-number generator available in statistical software and most graphing calculators. We assume the outcomes “have a boy” and “have a girl” are equally likely.

**Solution**

(a) We use MINITAB to perform the simulation. Set the seed in MINITAB to any value you wish, say 1970. Use the Integer Distribution* to generate random data that simulate three-child families. If we agree to let 0 represent a girl and 1 represent a boy, we can approximate the probability of having 2 boys by summing each row (adding up the number of boys), counting the number of 2s, and dividing by 100, the number of repetitions of the experiment. See Figure 4.

*bThe Integer Distribution involves a mathematical formula that uses a seed number to generate a sequence of equally likely random integers. Consult the technology manuals for setting the seed and generating sequences of integers.
Using MINITAB’s Tally command, we can determine the number of 2s that MINITAB randomly generated. See Figure 5.

**Figure 5**

<table>
<thead>
<tr>
<th>Tally for Discrete Variables: C4</th>
</tr>
</thead>
<tbody>
<tr>
<td>C4</td>
</tr>
<tr>
<td>-----</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>N = 100</td>
</tr>
</tbody>
</table>

Based on this figure, we approximate that there is a 32% probability that a three-child family will have 2 boys.

(b) Again, set the seed to 1970. Figure 6 shows the result of simulating 1,000 three-child families.

**Figure 6**

<table>
<thead>
<tr>
<th>Tally for Discrete Variables: C4</th>
</tr>
</thead>
<tbody>
<tr>
<td>C4</td>
</tr>
<tr>
<td>-----</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>N = 1000</td>
</tr>
</tbody>
</table>

We approximate that there is a 38.8% probability of a three-child family having 2 boys. Notice that more repetitions of the experiment (100 repetitions versus 1,000 repetitions) results in a probability closer to 37.5% as found in Example 7 (b).

5 Recognize and Interpret Subjective Probabilities

Suppose that a sports reporter is asked what he thinks the chances are for the Boston Red Sox to return to the World Series. The sports reporter will likely process information about the Red Sox (their pitching staff, lead-off hitter, and so on) and then come up with an educated guess of the likelihood. The reporter may respond that there is a 20% chance the Red Sox will return to the World Series. This forecast is a probability although it is not based on relative frequencies. We cannot, after all, repeat the experiment of playing a season under the same circumstances (same players, schedule, and so on) over and over. Nonetheless, the forecast of 20% does satisfy the criterion that a probability be between 0 and 1, inclusive. This forecast is known as a subjective probability.

**Definition**

A **subjective probability** of an outcome is a probability obtained on the basis of personal judgment.

It is important to understand that subjective probabilities are perfectly legitimate and are often the only method of assigning likelihood to an outcome. As another example, a financial reporter may ask an economist about the likelihood the economy will fall into recession next year. Again, we cannot conduct an experiment \( n \) times to obtain a relative frequency. The economist must use his or her knowledge of the current conditions of the economy and make an educated guess as to the likelihood of recession.
### 5.1 Assess Your Understanding

#### Concepts and Vocabulary

1. Describe the difference between classical and empirical probability.

2. What is the probability of an event that is impossible? Suppose that a probability is approximated to be zero based on empirical results. Does this mean the event is impossible? 
   - 0; no

3. In computing classical probabilities, all outcomes must be equally likely. Explain what this means.

4. What does it mean for an event to be unusual? Why should the cutoff for identifying unusual events not always be 0.05?

5. **True or False:** In a probability model, the sum of the probabilities of all outcomes must equal 1. 
   - True

6. **True or False:** Probability is a measure of the likelihood of a random phenomenon or chance behavior. 
   - True

7. In probability, a(n) **experiment** is any process that can be repeated in which the results are uncertain.

8. A(n) **event** is any collection of outcomes from a probability experiment.

9. Explain why probability can be considered a long-term relative frequency.

10. Explain the purpose of a tree diagram.

#### Skill Building

11. Verify that the following is a probability model. What do we call the outcome “blue”? 
   - Impossible event

12. Verify that the following is a probability model. If the model represents the colors of M&M’s in a bag of milk chocolate M&M’s, explain what the model implies. 
   - All of the M&M’s are yellow

13. Why is the following not a probability model? 
   - $P(\text{green}) < 0$

<table>
<thead>
<tr>
<th>Color</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>Red</td>
<td>0.3</td>
</tr>
<tr>
<td>Green</td>
<td>-0.3</td>
</tr>
<tr>
<td>Blue</td>
<td>0.2</td>
</tr>
<tr>
<td>Brown</td>
<td>0.4</td>
</tr>
<tr>
<td>Yellow</td>
<td>0.2</td>
</tr>
<tr>
<td>Orange</td>
<td>0.2</td>
</tr>
</tbody>
</table>

14. Why is the following not a probability model? 
   - Sum of probabilities not 1

<table>
<thead>
<tr>
<th>Color</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>Red</td>
<td>0.1</td>
</tr>
<tr>
<td>Green</td>
<td>0.1</td>
</tr>
<tr>
<td>Blue</td>
<td>0.1</td>
</tr>
<tr>
<td>Brown</td>
<td>0.4</td>
</tr>
<tr>
<td>Yellow</td>
<td>0.2</td>
</tr>
<tr>
<td>Orange</td>
<td>0.3</td>
</tr>
</tbody>
</table>

15. Which of the following numbers could be the probability of an event? 
   - 0, 0.01, 0.35, 1
   - 0, 0.01, 0.35, -0.4, 1, 1.4

16. Which of the following numbers could be the probability of an event? 
   - $1.5, \frac{1}{2}, \frac{3}{4}, 2, 0, -\frac{1}{4}, \frac{1}{2}, \frac{1}{4}, \frac{2}{3}, \frac{5}{2}, 0$

17. In five-card stud poker, a player is dealt five cards. The probability that the player is dealt two cards of the same value and three other cards of different value so that the player has a pair is 0.42. Explain what this probability means. If you play five-card stud 100 times, will you be dealt a pair exactly 42 times? Why or why not? 
   - No

18. In seven-card stud poker, a player is dealt seven cards. The probability that the player is dealt two cards of the same value and five other cards of different value so that the player has a pair is 0.44. Explain what this probability means. If you play seven-card stud 100 times, will you be dealt a pair exactly 42 times? Why or why not? 
   - No

19. Suppose that you toss a coin 100 times and get 95 heads and 5 tails. Based on these results, what is the estimated probability that the next flip results in a head? 
   - 0.95

20. Suppose that you roll a die 100 times and get six 80 times. Based on these results, what is the estimated probability that the next roll results in six? 
   - 0.8

21. Bob is asked to construct a probability model for rolling a pair of fair dice. He lists the outcomes as 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12. Because there are 11 outcomes, he reasoned, the probability of rolling a two must be $\frac{1}{11}$. What is wrong with Bob’s reasoning? 
   - Not equally likely outcomes
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22. **Blood Types** A person can have one of four blood types: A, B, AB, or O. If a person is randomly selected, is the probability they have blood type A equal to \( \frac{1}{4} \)? Why? No

23. If a person rolls a six-sided die and then flips a coin, describe the sample space of possible outcomes using 1, 2, 3, 4, 5, 6 for the die outcomes and H, T for the coin outcomes.

24. If a basketball player shoots three free throws, describe the sample space of possible outcomes using S for a made free throw and F for a missed free throw.

25. According to the U.S. Department of Education, 42.8% of 3-year-olds are enrolled in day care. What is the probability that a randomly selected 3-year-old is enrolled in day care? 0.428

26. According to the American Veterinary Medical Association, the proportion of households owning a dog is 0.372. What is the probability that a randomly selected household owns a dog? 0.372

For Problems 27–30, let the sample space be \( S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \). Suppose the outcomes are equally likely.

27. Compute the probability of the event \( E = \{1, 2, 3\} \).

28. Compute the probability of the event \( F = \{3, 5, 9, 10\} \).

29. Compute the probability of the event \( E = \{\text{an even number less than 9}\} \).

30. Compute the probability of the event \( F = \{\text{an odd number}\} \). 27. \( \frac{3}{10} \) 28. \( \frac{2}{5} \) 29. \( \frac{2}{5} \) 30. \( \frac{1}{2} \)

**Applying the Concepts**

31. **Play Sports?** A survey of 500 randomly selected high school students determined that 288 played organized sports.

(a) What is the probability that a randomly selected high school student plays organized sports? 0.576

(b) Interpret this probability.

32. **Volunteer?** In a survey of 1,100 female adults (18 years of age or older), it was determined that 341 volunteered at least once in the past year.

(a) What is the probability that a randomly selected adult female volunteered at least once in the past year? 0.31

(b) Interpret this probability.

33. **Planting Tulips** A bag of 100 tulip bulbs purchased from a nursery contains 40 red tulip bulbs, 35 yellow tulip bulbs, and 25 purple tulip bulbs.

(a) What is the probability that a randomly selected tulip bulb is red? 0.4

(b) What is the probability that a randomly selected tulip bulb is purple? 0.25

(c) Interpret these two probabilities.

34. **Golf Balls** The local golf store sells an “onion bag” that contains 80 “experienced” golf balls. Suppose the bag contains 35 Titleists, 25 Maxflis, and 20 Top-Flites.

(a) What is the probability that a randomly selected golf ball is a Titleist? 0.4375

(b) What is the probability that a randomly selected golf ball is a Top-Flite? 0.25

(c) Interpret these two probabilities.

35. **Roulette** In the game of roulette, a wheel consists of 38 slots numbered 0, 00, 1, 2, \ldots, 36. (See the photo.) To play the game, a metal ball is spun around the wheel and is allowed to fall into one of the numbered slots.

(a) Determine the sample space. \( \{0, 00, 1, 2, \ldots, 36\} \)

(b) Determine the probability that the metal ball falls into the slot marked 8. Interpret this probability.

(c) Determine the probability that the metal ball lands in an odd slot. Interpret this probability.

36. **Birthdays** Exclude leap years from the following calculations and assume each birthday is equally likely.

(a) Determine the probability that a randomly selected person has a birthday on the 1st day of a month. Interpret this probability.

(b) Determine the probability that a randomly selected person has a birthday on the 31st day of a month. Interpret this probability.

(c) Determine the probability that a randomly selected person was born in December. Interpret this probability.

(d) Determine the probability that a randomly selected person has a birthday on November 8. Interpret this probability.

(e) If you just met somebody and she asked you to guess her birthday, are you likely to be correct? No

(f) Do you think it is appropriate to use the methods of classical probability to compute the probability that a person is born in December?

37. **Genetics** A gene is composed of two alleles. An allele can be either dominant or recessive. Suppose that a husband and wife, who are both carriers of the sickle-cell anemia allele but do not have the disease, decide to have a child. Because both parents are carriers of the disease, each has one dominant normal-cell allele (S) and one recessive sickle-cell allele (s). Therefore, the genotype of each parent is Ss. Each parent contributes one allele to his or her offspring, with each allele being equally likely. 37. (a) \{SS, Ss, sS, ss\}

(a) List the possible genotypes of their offspring.

(b) What is the probability that the offspring will have sickle-cell anemia? In other words, what is the probability that the offspring will have genotype ss? Interpret this probability.

(c) What is the probability that the offspring will not have sickle-cell anemia but will be a carrier? In other words, what is the probability that the offspring will have one dominant normal-cell allele and one recessive sickle-cell allele? Interpret this probability.
38. **More Genetics** In Problem 37, we learned that for some diseases, such as sickle-cell anemia, an individual will get the disease only if he or she receives both recessive alleles. This is not always the case. For example, Huntington’s disease only requires one dominant gene for an individual to contract the disease. Suppose that a husband and wife, who both have a dominant Huntington’s disease allele \( S \) and a normal recessive allele \( s \), decide to have a child.

(a) List the possible genotypes of their offspring.
(b) What is the probability that the offspring will not have Huntington’s disease? In other words, what is the probability that the offspring will have genotype \( ss \)? Interpret this probability.
(c) What is the probability that the offspring will have Huntington’s disease?

39. **College Survey** In a national survey conducted by the Centers for Disease Control to determine college students’ health-risk behaviors, college students were asked, “How often do you wear a seatbelt when riding in a car driven by someone else?” The frequencies appear in the following table:

<table>
<thead>
<tr>
<th>Response</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>Never</td>
<td>125</td>
</tr>
<tr>
<td>Rarely</td>
<td>324</td>
</tr>
<tr>
<td>Sometimes</td>
<td>552</td>
</tr>
<tr>
<td>Most of the time</td>
<td>1,257</td>
</tr>
<tr>
<td>Always</td>
<td>2,518</td>
</tr>
</tbody>
</table>

(a) Construct a probability model for seatbelt use by a passenger.
(b) Would you consider it unusual to find a college student who never wears a seatbelt when riding in a car driven by someone else? Why? Yes

40. **College Survey** In a national survey conducted by the Centers for Disease Control to determine college students’ health-risk behaviors, college students were asked, “How often do you wear a seatbelt when driving a car?” The frequencies appear in the following table:

<table>
<thead>
<tr>
<th>Response</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>Never</td>
<td>118</td>
</tr>
<tr>
<td>Rarely</td>
<td>249</td>
</tr>
<tr>
<td>Sometimes</td>
<td>345</td>
</tr>
<tr>
<td>Most of the time</td>
<td>716</td>
</tr>
<tr>
<td>Always</td>
<td>3,093</td>
</tr>
</tbody>
</table>

(a) Construct a probability model for seatbelt use by a driver.
(b) Is it unusual for a college student to never wear a seatbelt when driving a car? Why? Yes

41. **Larceny Theft** A police officer randomly selected 642 police records of larceny thefts. The following data represent the number of offenses for various types of larceny thefts.

(a) Construct a probability model for type of larceny theft.
(b) Are purse snatching larcenies unusual? Yes
(c) Are bicycle larcenies unusual? No

Type of Larceny Theft

<table>
<thead>
<tr>
<th>Number of Offenses</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pocket picking</td>
</tr>
<tr>
<td>Purse snatching</td>
</tr>
<tr>
<td>Shoplifting</td>
</tr>
<tr>
<td>From motor vehicles</td>
</tr>
<tr>
<td>Motor vehicle accessories</td>
</tr>
<tr>
<td>Bicycles</td>
</tr>
<tr>
<td>From buildings</td>
</tr>
<tr>
<td>From coin-operated machines</td>
</tr>
</tbody>
</table>

Source: U.S. Federal Bureau of Investigation

42. **Multiple Births** The following data represent the number of live multiple-delivery births (three or more babies) in 2005 for women 15 to 44 years old.

<table>
<thead>
<tr>
<th>Age</th>
<th>Number of Multiple Births</th>
</tr>
</thead>
<tbody>
<tr>
<td>15–19</td>
<td>83</td>
</tr>
<tr>
<td>20–24</td>
<td>465</td>
</tr>
<tr>
<td>25–29</td>
<td>1,635</td>
</tr>
<tr>
<td>30–34</td>
<td>2,443</td>
</tr>
<tr>
<td>35–39</td>
<td>1,604</td>
</tr>
<tr>
<td>40–44</td>
<td>344</td>
</tr>
</tbody>
</table>


(a) Construct a probability model for number of multiple births.
(b) In the sample space of all multiple births, are multiple births for 15- to 19-year-old mothers unusual? Yes
(c) In the sample space of all multiple births, are multiple births for 40- to 44-year-old mothers unusual? Yes

42. (c) Not too unusual

Problems 43–46 use the given table, which lists six possible assignments of probabilities for tossing a coin twice, to answer the following questions.

<table>
<thead>
<tr>
<th>Assignments</th>
<th>Sample Space</th>
</tr>
</thead>
<tbody>
<tr>
<td>HH</td>
<td>1 1 1 1</td>
</tr>
<tr>
<td>HT</td>
<td>1 1 1 1</td>
</tr>
<tr>
<td>TH</td>
<td>1 1 1 1</td>
</tr>
<tr>
<td>TT</td>
<td>1 1 1 1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Assignments</th>
<th>Sample Space</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1 1 1 1</td>
</tr>
<tr>
<td>B</td>
<td>0 0 0 1</td>
</tr>
<tr>
<td>C</td>
<td>1 1 1 1</td>
</tr>
<tr>
<td>D</td>
<td>1 1 1 1</td>
</tr>
<tr>
<td>E</td>
<td>1 1 1 1</td>
</tr>
<tr>
<td>F</td>
<td>1 1 1 1</td>
</tr>
</tbody>
</table>

43. Which of the assignments of probabilities are consistent with the definition of a probability model? A, B, C, F
44. Which of the assignments of probabilities should be used if the coin is known to be fair? A
45. Which of the assignments of probabilities should be used if the coin is known to always come up tails?  B

46. Which of the assignments of probabilities should be used if tails is twice as likely to occur as heads?  F

47. **Going to Disney World**  John, Roberto, Clarice, Dominique, and Marco work for a publishing company. The company wants to send two employees to a statistics conference in Orlando. To be fair, the company decides that the two individuals who get to attend will have their names randomly drawn from a hat.

(a) Determine the sample space of the experiment. That is, list all possible simple random samples of size \( n = 2 \).
(b) What is the probability that Clarice and Dominique attend the conference?  0.1
(c) What is the probability that Clarice attends the conference?  0.4
(d) What is the probability that John stays home?  0.6

48. **Six Flags**  In 2008, Six Flags St. Louis had eight roller coasters: The Screamin’ Eagle, The Boss, River King Mine Train, Batman the Ride, Mr. Freeze, Ninja, Tony Hawk’s Big Spin, and Evel Knievel. Of these, The Boss, The Screamin’ Eagle, and Evel Knievel are wooden coasters. Ethan wants to ride two more roller coasters before leaving the park (not the same one twice) and decides to select them by drawing names from a hat.

(a) Determine the sample space of the experiment. That is, list all possible simple random samples of size \( n = 2 \).
(b) What is the probability that Ethan will ride Mr. Freeze and Evel Knievel?  0.036
(c) What is the probability that Ethan will ride the Screamin’ Eagle?  0.25
(d) What is the probability that Ethan will ride two wooden roller coasters?  0.107
(e) What is the probability that Ethan will not ride any wooden roller coasters?  0.357

49. **Barry Bonds**  On October 5, 2001, Barry Bonds broke Mark McGwire's home-run record for a single season by hitting his 71st and 72nd home runs. Bonds went on to hit one more home run before the season ended, for a total of 73. Of the 73 home runs, 24 went to right field, 26 went to right center field, 11 went to center field, 10 went to left center field, and 2 went to left field.

Source: Baseball-almanac.com

(a) What is the probability that a randomly selected home run was hit to right field?
(b) What is the probability that a randomly selected home run was hit to left field?
(c) Was it unusual for Barry Bonds to hit a home run to left field? Explain.  Yes

50. **Rolling a Die**

(a) Roll a single die 50 times, recording the result of each roll of the die. Use the results to approximate the probability of rolling a three.
(b) Roll a single die 100 times, recording the result of each roll of the die. Use the results to approximate the probability of rolling a three.
(c) Compare the results of (a) and (b) to the classical probability of rolling a three.

51. **Simulation**  Use a graphing calculator or statistical software to simulate rolling a six-sided die 100 times, using an integer distribution with numbers one through six.

(a) Use the results of the simulation to compute the probability of rolling a one.
(b) Repeat the simulation. Compute the probability of rolling a one.
(c) Simulate rolling a six-sided die 500 times. Compute the probability of rolling a one.
(d) Which simulation resulted in the closest estimate to the probability that would be obtained using the classical method?

52. **Classifying Probability**  Determine whether the following probabilities are computed using classical methods, empirical methods, or subjective methods.

(a) The probability of having eight girls in an eight-child family is 0.390625%.  Classical
(b) On the basis of a survey of 1,000 families with eight children, the probability of a family having eight girls is 0.54%.  Empirical
(c) According to a sports analyst, the probability that the Chicago Bears will win their next game is about 30%.  Subjective
(d) On the basis of clinical trials, the probability of efficacy of a new drug is 75%.  Classical

53. **Checking for Loaded Dice**  You suspect a pair of dice to be loaded and conduct a probability experiment by rolling each die 400 times. The outcome of the experiment is listed in the following table:

<table>
<thead>
<tr>
<th>Value of Die</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>105</td>
</tr>
<tr>
<td>2</td>
<td>47</td>
</tr>
<tr>
<td>3</td>
<td>44</td>
</tr>
<tr>
<td>4</td>
<td>49</td>
</tr>
<tr>
<td>5</td>
<td>51</td>
</tr>
<tr>
<td>6</td>
<td>104</td>
</tr>
</tbody>
</table>

Do you think the dice are loaded? Why?  Yes

54. Conduct a survey in your school by randomly asking 50 students whether they drive to school. Based on the results of the survey, approximate the probability that a randomly selected student drives to school.

55. In 2006, the median income of families in the United States was $58,500. What is the probability that a randomly selected family has an income greater than $58,500?  0.5

56. The middle 50% of enrolled freshmen at Washington University in St. Louis had SAT math scores in the range 700–780. What is the probability that a randomly selected freshman at Washington University has a SAT math score of 700 or higher?  0.75

57. **The Probability Applet**  Load the long-run probability applet on your computer.

(a) Choose the “simulating the probability of a head with a fair coin” applet and simulate flipping a fair coin 10 times. What is the estimated probability of a head based on these 10 trials?
(b) Reset the applet. Simulate flipping a fair coin 10 times a second time. What is the estimated probability of a
section 5.1 Probability Rules

58. Putting It Together: Drug Side Effects In placebo-controlled clinical trials for the drug Viagra, 734 subjects received Viagra and 725 subjects received a placebo (subjects did not know which treatment they received). The following table summarizes reports of various side effects that were reported.  
(a) Is the variable “adverse effect” qualitative or quantitative? Qualitative  
(b) Which type of graph would be appropriate to display the information in the table? Construct the graph. Side-by-side relative frequency bar graph

<table>
<thead>
<tr>
<th>Adverse Effect</th>
<th>Viagra (n = 734)</th>
<th>Placebo (n = 725)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Headache</td>
<td>117</td>
<td>29</td>
</tr>
<tr>
<td>Flushing</td>
<td>73</td>
<td>7</td>
</tr>
<tr>
<td>Dyspepsia</td>
<td>51</td>
<td>15</td>
</tr>
<tr>
<td>Nasal congestion</td>
<td>29</td>
<td>15</td>
</tr>
<tr>
<td>Urinary tract infection</td>
<td>22</td>
<td>15</td>
</tr>
<tr>
<td>Abnormal vision</td>
<td>22</td>
<td>0</td>
</tr>
<tr>
<td>Diarrhea</td>
<td>22</td>
<td>7</td>
</tr>
<tr>
<td>Dizziness</td>
<td>15</td>
<td>7</td>
</tr>
<tr>
<td>Rash</td>
<td>15</td>
<td>7</td>
</tr>
</tbody>
</table>

(c) What is the estimated probability that a randomly selected subject from the Viagra group reported experiencing flushing? Would this be unusual? 0.099; no  
(d) What is the estimated probability that a subject receiving a placebo would report experiencing flushing? Would this be unusual? 0.010; yes  
(e) If a subject reports flushing after receiving a treatment, what might you conclude? They received the Viagra treatment  
(f) What type of experimental design is this? Completely randomized design

TECHNOLOGY STEP-BY-STEP

TI-83/84 Plus  
1. Set the seed by entering any number on the HOME screen. Press the STO button, press the MATH button, highlight the PRB menu, and highlight 1 : rand and hit ENTER. With the cursor on the HOME screen, hit ENTER.  
2. Press the MATH button and highlight the PRB menu. Highlight 5:randInt ( and hit ENTER.  
3. After the randInt ( on the HOME screen, type 1, n, number of repetitions of experiment ), where n is the number of equally likely outcomes. For example, to simulate rolling a single die 50 times, we type \( \text{randInt}(1, 6, 50) \)  
4. Press the STO button and then 2nd 1, and hit ENTER to store the data in L1.  
5. Draw a histogram of the data using the outcomes as classes. TRACE to obtain outcomes.

MINITAB  
1. Set the seed by selecting the Calc menu and highlighting Set Base. . . . Insert any seed you wish into the cell and click OK.

Simulation  
2. Select the Calc menu, highlight Random Data, and then highlight Integer. To simulate rolling a single die 100 times, fill in the window as shown in Figure 4 on page 267.  
3. Select the Stat menu, highlight Tables, and then highlight Tally. . . . Enter C1 into the variables cell. Make sure that the Counts box is checked and click OK.

Excel  
1. With cell A1 selected, press the fx button.  
2. Highlight Math & Trig in the Function category window. Then highlight RANDBETWEEN in the Function Name: window. Click OK.  
3. To simulate rolling a die 50 times, enter 1 for the lower limit and 6 for the upper limit. Click OK.  
4. Copy the contents of cell A1 into cells A2 through A50.
5.2 THE ADDITION RULE AND COMPLEMENTS

Preparation for this Section Before getting started, review the following:

- Contingency Tables (Section 4.4, p. 239)

**Objectives**

1. Use the Addition Rule for disjoint events
2. Use the General Addition Rule
3. Compute the probability of an event using the Complement Rule

**Note to Instructor**

If you like, you can print out and distribute the Preparing for This Section quiz located in the Instructor’s Resource Center. The purpose of the quiz is to verify that the students have the prerequisite knowledge for the section.

**In Other Words**

Two events are disjoint if they cannot occur at the same time.

**Definition**

Two events are disjoint if they have no outcomes in common. Another name for disjoint events is mutually exclusive events.

It is often helpful to draw pictures of events. Such pictures, called Venn diagrams, represent events as circles enclosed in a rectangle. The rectangle represents the sample space, and each circle represents an event. For example, suppose we randomly select chips from a bag. Each chip is labeled 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. Let E represent the event “choose a number less than or equal to 2,” and let F represent the event “choose a number greater than or equal to 8.” Because E and F do not have any outcomes in common, they are disjoint. Figure 7 shows a Venn diagram of these disjoint events.

![Figure 7](image)

Notice that the outcomes in event E are inside circle E, and the outcomes in event F are inside the circle F. All outcomes in the sample space that are not in E or F are outside the circles, but inside the rectangle. From this diagram, we know that

\[ P(E) = \frac{N(E)}{N(S)} = \frac{3}{10} = 0.3 \text{ and } P(F) = \frac{N(F)}{N(S)} = \frac{2}{10} = 0.2. \]

In addition, \( P(E \text{ or } F) = \frac{N(E \text{ or } F)}{N(S)} = \frac{5}{10} = 0.5 \) and \( P(E \text{ or } F) = P(E) + P(F) = 0.3 + 0.2 = 0.5. \) This result occurs because of the Addition Rule for Disjoint Events.

The Addition Rule for Disjoint Events states that, if you have two events that have no outcomes in common, the probability that one or the other occurs is the sum of their probabilities.

**Addition Rule for Disjoint Events**

If E and F are disjoint (or mutually exclusive) events, then

\[ P(E \text{ or } F) = P(E) + P(F) \]
Let event $G$ represent “the number is a 5 or 6.” The Venn diagram in Figure 8 illustrates the Addition Rule for more than two disjoint events using the chip example. Notice that no pair of events has any outcomes in common. So, from the Venn diagram, we can see that $P(E) = \frac{N(E)}{N(S)} = \frac{3}{10} = 0.3$, $P(F) = \frac{N(F)}{N(S)} = \frac{2}{10} = 0.2$, and $P(G) = \frac{N(G)}{N(S)} = \frac{2}{10} = 0.2$. In addition, $P(E \text{ or } F \text{ or } G) = P(E) + P(F) + P(G) = 0.3 + 0.2 + 0.2 = 0.7$.

**EXAMPLE 1**

**Benford’s Law and the Addition Rule for Disjoint Events**

**Problem:** Our number system consists of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. Because we do not write numbers such as 12 as 012, the first significant digit in any number must be 1, 2, 3, 4, 5, 6, 7, 8, or 9. Although we may think that each digit appears with equal frequency so that each digit has a $\frac{1}{9}$ probability of being the first significant digit, this is, in fact, not true. In 1881, Simon Necomb discovered that digits do not occur with equal frequency. This same result was discovered again in 1938 by physicist Frank Benford. After studying lots and lots of data, he was able to assign probabilities of occurrence for each of the first digits, as shown in Table 5.

<table>
<thead>
<tr>
<th>Digit</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.301</td>
</tr>
<tr>
<td>2</td>
<td>0.176</td>
</tr>
<tr>
<td>3</td>
<td>0.125</td>
</tr>
<tr>
<td>4</td>
<td>0.097</td>
</tr>
<tr>
<td>5</td>
<td>0.079</td>
</tr>
<tr>
<td>6</td>
<td>0.067</td>
</tr>
<tr>
<td>7</td>
<td>0.058</td>
</tr>
<tr>
<td>8</td>
<td>0.051</td>
</tr>
<tr>
<td>9</td>
<td>0.046</td>
</tr>
</tbody>
</table>


The probability model is now known as *Benford’s Law* and plays a major role in identifying fraudulent data on tax returns and accounting books.

(a) Verify that Benford’s Law is a probability model.

(b) Use Benford’s Law to determine the probability that a randomly selected first digit is 1 or 2.

(c) Use Benford’s Law to determine the probability that a randomly selected first digit is at least 6.

**Approach:** For part (a), we need to verify that each probability is between 0 and 1 and that the sum of all probabilities equals 1. For parts (b) and (c), we use the Addition Rule for Disjoint Events.

**Solution**

(a) In looking at Table 5, we see that each probability is between 0 and 1. In addition, the sum of all the probabilities is 1.

$$0.301 + 0.176 + 0.125 + \cdots + 0.046 = 1$$

Because rules 1 and 2 are satisfied, Table 5 represents a probability model.
(b) \[ P(1 \text{ or } 2) = P(1) + P(2) = 0.301 + 0.176 = 0.477 \]

If we looked at 100 numbers, we would expect about 48 to begin with 1 or 2.

(c) \[ P(\text{at least 6}) = P(6 \text{ or } 7 \text{ or } 8 \text{ or } 9) = P(6) + P(7) + P(8) + P(9) = 0.067 + 0.058 + 0.051 + 0.046 = 0.222 \]

If we looked at 100 numbers, we would expect about 22 to begin with 6, 7, 8, or 9.

**EXAMPLE 2** A Deck of Cards and the Addition Rule for Disjoint Events

**Problem:** Suppose that a single card is selected from a standard 52-card deck, such as the one shown in Figure 9.

**Figure 9**

(a) Compute the probability of the event \( E = \) “drawing a king.”

(b) Compute the probability of the event \( E = \) “drawing a king” or \( F = \) “drawing a queen.”

(c) Compute the probability of the event \( E = \) “drawing a king” or \( F = \) “drawing a queen” or \( G = \) “drawing a jack.”

**Approach:** We will use the classical method for computing the probabilities because the outcomes are equally likely and easy to count. We use the Addition Rule for Disjoint Events to compute the probabilities in parts (b) and (c) because the events are mutually exclusive. For example, you cannot simultaneously draw a king and a queen.

**Solution:** The sample space consists of the 52 cards in the deck, so \( N(S) = 52 \).

(a) A standard deck of cards has four kings, so \( N(E) = 4 \). Therefore,

\[ P(\text{king}) = P(E) = \frac{N(E)}{N(S)} = \frac{4}{52} = \frac{1}{13} \]

(b) A standard deck of cards also has four queens. Because events \( E \) and \( F \) are mutually exclusive, we use the Addition Rule for Disjoint Events. So

\[ P(\text{king or queen}) = P(\text{or } F) = P(E) + P(F) = \frac{4}{52} + \frac{4}{52} = \frac{8}{52} = \frac{2}{13} \]
(c) Because events $E$, $F$, and $G$ are mutually exclusive, we use the Addition Rule for Disjoint Events extended to two or more disjoint events. So

$$P(\text{king or queen or jack}) = P(E \text{ or } F \text{ or } G) = P(E) + P(F) + P(G) = \frac{4}{52} + \frac{4}{52} + \frac{4}{52} = \frac{12}{52} = \frac{3}{13}$$

**Use the General Addition Rule**

A question that you may be asking yourself is, “What if I need to compute the probability of two events that are not disjoint?”

Consider the chip example. Suppose that we are randomly selecting chips from a bag. Each chip is labeled 0, 1, 2, 3, 4, 5, 6, 7, 8, or 9. Let $E$ represent the event “choose an odd number,” and let $F$ represent the event “choose a number less than or equal to 4.” Because $E = \{1, 3, 5, 7, 9\}$ and $F = \{0, 1, 2, 3, 4\}$ have the outcomes 1 and 3 in common, the events are not disjoint. Figure 10 shows a Venn diagram of these events.

![Venn diagram](image)

We can compute $P(E \text{ or } F)$ directly by counting because each outcome is equally likely. There are 8 outcomes in $E$ or $F$ and 10 outcomes in the sample space, so

$$P(E \text{ or } F) = \frac{N(E \text{ or } F)}{N(S)} = \frac{8}{10} = \frac{4}{5}$$

If we attempt to compute $P(E \text{ or } F)$ using the Addition Rule for Disjoint Events, we obtain the following:

$$P(E \text{ or } F) = P(E) + P(F) = \frac{5}{10} + \frac{5}{10} = \frac{10}{10} = 1$$

This implies that the chips labeled 6 and 8 will never be selected, which contradicts our assumption that all the outcomes are equally likely. Our result is incorrect because we counted the outcomes 1 and 3 twice: once for event $E$ and once for event $F$. To avoid this double counting, we have to subtract the probability corresponding to the overlapping region, $E$ and $F$. That is, we have to subtract $P(E \text{ and } F) = \frac{2}{10}$ from the result and obtain

$$P(E \text{ or } F) = P(E) + P(F) - P(E \text{ and } F) = \frac{5}{10} + \frac{5}{10} - \frac{2}{10} = \frac{8}{10} = \frac{4}{5}$$

**Note to Instructor**

If you wish to design your course so that it minimizes probability coverage, the material in Objective 2 may be skipped without loss of continuity.
which agrees with the result we obtained by counting. These results can be generalized in the following rule:

**The General Addition Rule**  
For any two events $E$ and $F$,  
$$P(E \text{ or } F) = P(E) + P(F) - P(E \text{ and } F)$$

**EXAMPLE 3**  
**Computing Probabilities for Events That Are Not Disjoint**

Problem: Suppose that a single card is selected from a standard 52-card deck. Compute the probability of the event $E = \text{“drawing a king”}$ or $H = \text{“drawing a diamond.”}$

Approach: The events are not disjoint because the outcome “king of diamonds” is in both events, so we use the General Addition Rule.

Solution

$$P(\text{king or diamond}) = P(\text{king}) + P(\text{diamond}) - P(\text{king of diamonds})$$

$$= \frac{4}{52} + \frac{13}{52} - \frac{1}{52}$$

$$= \frac{16}{52} = \frac{4}{13}$$

Now Work Problem 31

Consider the data shown in Table 6, which represent the marital status of males and females 18 years old or older in the United States in 2006.

<table>
<thead>
<tr>
<th>Table 6</th>
<th>Males (in millions)</th>
<th>Females (in millions)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Never married</td>
<td>30.3</td>
<td>25.0</td>
</tr>
<tr>
<td>Married</td>
<td>63.6</td>
<td>64.1</td>
</tr>
<tr>
<td>Widowed</td>
<td>2.6</td>
<td>11.3</td>
</tr>
<tr>
<td>Divorced</td>
<td>9.7</td>
<td>13.1</td>
</tr>
</tbody>
</table>

Source: U.S. Census Bureau, *Current Population Reports*

Table 6 is called a **contingency table** or **two-way table**, because it relates two categories of data. The **row variable** is marital status, because each row in the table describes the marital status of each individual. The **column variable** is gender. Each box inside the table is called a **cell**. For example, the cell corresponding to married individuals who are male is in the second row, first column. Each cell contains the frequency of the category: There were 63.6 million married males in the United States in 2006. Put another way, in the United States in 2006, there were 63.6 million individuals who were male and married.

**EXAMPLE 4**  
**Using the Addition Rule with Contingency Tables**

Problem: Using the data in Table 6,

(a) Determine the probability that a randomly selected U.S. resident 18 years old or older is male.

(b) Determine the probability that a randomly selected U.S. resident 18 years old or older is widowed.
(c) Determine the probability that a randomly selected U.S. resident 18 years old or older is widowed or divorced.

(d) Determine the probability that a randomly selected U.S. resident 18 years old or older is male or widowed.

Approach: We first add up the entries in each row and column so that we get the total number of people in each category. We can then determine the probabilities using either the Addition Rule for Disjoint Events or the General Addition Rule.

Solution: Add the entries in each column. For example, in the “male” column we find that there are 30.3 + 63.6 + 2.6 + 9.7 = 106.2 million males 18 years old or older in the United States. Add the entries in each row. For example, in the “never married” row we find there are 30.3 + 25.0 = 55.3 million U.S. residents 18 years old or older who have never married. Adding the row totals or column totals, we find there are 106.2 + 113.5 = 55.3 + 127.7 + 13.9 + 22.8 = 219.7 million U.S. residents 18 years old or older.

(a) There are 106.2 million males 18 years old or older and 219.7 million U.S. residents 18 years old or older. The probability that a randomly selected U.S. resident 18 years old or older is male is

\[
\frac{106.2}{219.7} = 0.483.
\]

(b) There are 13.9 million U.S. residents 18 years old or older who are widowed. The probability that a randomly selected U.S. resident 18 years old or older is widowed is

\[
\frac{13.9}{219.7} = 0.063.
\]

(c) The events widowed and divorced are disjoint. Do you see why? We use the Addition Rule for Disjoint Events.

\[
P(\text{widowed or divorced}) = P(\text{widowed}) + P(\text{divorced})
\]

\[
= \frac{13.9}{219.7} + \frac{22.8}{219.7} = \frac{36.7}{219.7}
\]

\[
= 0.167
\]

(d) The events male and widowed are not mutually exclusive. In fact, there are 2.6 million males who are widowed in the United States. Therefore, we use the General Addition Rule to compute \(P(\text{male or widowed})\):

\[
P(\text{male or widowed}) = P(\text{male}) + P(\text{widowed}) - P(\text{male and widowed})
\]

\[
= \frac{106.2}{219.7} + \frac{13.9}{219.7} - \frac{2.6}{219.7}
\]

\[
= \frac{117.5}{219.7} = 0.535
\]

### Compute the Probability of an Event Using the Complement Rule

Suppose that the probability of an event \(E\) is known and we would like to determine the probability that \(E\) does not occur. This can easily be accomplished using the idea of complements.

**Definition**

**Complement of an Event**

Let \(S\) denote the sample space of a probability experiment and let \(E\) denote an event. The **complement of \(E\)**, denoted \(E'\), is all outcomes in the sample space \(S\) that are not outcomes in the event \(E\).
Because $E$ and $E^c$ are mutually exclusive,

$$P(E \text{ or } E^c) = P(E) + P(E^c) = P(S) = 1$$

Subtracting $P(E)$ from both sides, we obtain

$$P(E^c) = 1 - P(E)$$

We have the following result.

**Complement Rule**

If $E$ represents any event and $E^c$ represents the complement of $E$, then

$$P(E^c) = 1 - P(E)$$

Figure 11 illustrates the Complement Rule using a Venn diagram.

**EXAMPLE 5**

**Computing Probabilities Using Complements**

**Problem:** According to the National Gambling Impact Study Commission, 52% of Americans have played state lotteries. What is the probability that a randomly selected American has not played a state lottery?

**Approach:** Not playing a state lottery is the complement of playing a state lottery. We compute the probability using the Complement Rule.

**Solution**

$$P(\text{not played state lottery}) = 1 - P(\text{played state lottery}) = 1 - 0.52 = 0.48$$

There is a 48% probability of randomly selecting an American who has not played a state lottery.

**EXAMPLE 6**

**Computing Probabilities Using Complements**

**Problem:** The data in Table 7 represent the income distribution of households in the United States in 2006.

<table>
<thead>
<tr>
<th>Annual Income</th>
<th>Number (in thousands)</th>
<th>Annual Income</th>
<th>Number (in thousands)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Less than $10,000</td>
<td>8,899</td>
<td>$50,000 to $74,999</td>
<td>21,222</td>
</tr>
<tr>
<td>$10,000 to $14,999</td>
<td>6,640</td>
<td>$75,000 to $99,999</td>
<td>13,215</td>
</tr>
<tr>
<td>$15,000 to $24,999</td>
<td>12,722</td>
<td>$100,000 to $149,999</td>
<td>12,164</td>
</tr>
<tr>
<td>$25,000 to $34,999</td>
<td>12,447</td>
<td>$150,000 to $199,999</td>
<td>3,981</td>
</tr>
<tr>
<td>$35,000 to $49,999</td>
<td>16,511</td>
<td>$200,000 or more</td>
<td>3,817</td>
</tr>
</tbody>
</table>

*Source: U.S. Census Bureau*
(a) Compute the probability that a randomly selected household earned $200,000 or more in 2006.
(b) Compute the probability that a randomly selected household earned less than $200,000 in 2006.
(c) Compute the probability that a randomly selected household earned at least $10,000 in 2006.

**Approach:** The probabilities will be determined by finding the relative frequency of each event. We have to find the total number of households in the United States in 2006.

**Solution**

(a) There were a total of 8,899 thousand households in the United States in 2006 and 3,817 thousand of them earned $200,000 or more. The probability that a randomly selected household in the United States earned $200,000 or more in 2006 is

\[ P(\text{less than } 200,000) = 1 - P(\text{less than } 200,000) = 0.966 \]

There is a 96.6% probability of randomly selecting a household that earned less than $200,000 in 2006.

(b) We could compute the probability of randomly selecting a household that earned less than $200,000 in 2006 by adding the relative frequencies of each category less than $200,000, but it is easier to use complements. The complement of earning less than $200,000 is earning $200,000 or more. Therefore,

\[ P(\text{at least } 10,000) = 1 - P(\text{less than } 10,000) = 0.920 \]

There is a 92.0% probability of randomly selecting a household that earned at least $10,000 in 2006.

5.2 **ASSess YOUR UNDERSTANDING**

**Concepts and Vocabulary**

1. What does it mean when two events are disjoint?
2. If \( E \) and \( F \) are disjoint events, then \( P(E \text{ or } F) = \) ________.
3. If \( E \) and \( F \) are not disjoint events, then \( P(E \text{ or } F) = \) ________.
4. What does it mean when two events are complements?

**Skill Building**

In Problems 5–12, a probability experiment is conducted in which the sample space of the experiment is

\[ S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\} \]

Let event \( E = \{2, 3, 4, 5, 6, 7\} \), event \( F = \{5, 6, 7, 8, 9\} \), event \( G = \{9, 10, 11, 12\} \), and event \( H = \{2, 3, 4\} \). Assume that each outcome is equally likely.

5. List the outcomes in \( E \) and \( F \). Are \( E \) and \( F \) mutually exclusive? \( \{5, 6, 7\}; \text{ no} \)
6. List the outcomes in \( F \) and \( G \). Are \( F \) and \( G \) mutually exclusive? \( \{9\}; \text{ no} \)
7. List the outcomes in \( F \) or \( G \). Now find \( P(F \text{ or } G) \) by counting the number of outcomes in \( F \) or \( G \). Determine \( P(F \text{ or } G) \) using the General Addition Rule. \( \{5, 6, 7, 8, 9, 10, 11, 12\}; \frac{7}{12} \)
8. List the outcomes in \( E \) or \( H \). Now find \( P(E \text{ or } H) \) by counting the number of outcomes in \( E \) or \( H \). Determine \( P(E \text{ or } H) \) using the General Addition Rule. \( \{2, 3, 4, 5, 6, 7\}; \frac{1}{2} \)
9. List the outcomes in \( E \) and \( G \). Are \( E \) and \( G \) mutually exclusive? \( \{\}; \text{ yes} \)
10. List the outcomes in \( F \) and \( H \). Are \( F \) and \( H \) mutually exclusive? \( \{\}; \text{ yes} \)
11. List the outcomes in \( E^c \). Find \( P(E^c) \). \( \{1, 8, 9, 10, 11, 12\}; \frac{7}{12} \)
12. List the outcomes in \( F^c \). Find \( P(F^c) \). \( \{1, 2, 3, 4, 10, 11, 12\}; \frac{7}{12} \)
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In Problems 13–18, find the probability of the indicated event if
\[ P(E) = 0.25 \text{ and } P(F) = 0.45. \]
13. Find \( P(E \text{ or } F) \) if \( P(E \text{ and } F) = 0.15 \). \( 0.55 \)
14. Find \( P(E \text{ and } F) \) if \( P(E \text{ or } F) = 0.6 \). \( 0.1 \)
15. Find \( P(E \text{ or } F) \) if \( E \) and \( F \) are mutually exclusive. \( 0.7 \)
16. Find \( P(E \text{ or } F) \) if \( E \) and \( F \) are mutually exclusive. \( 0 \)
17. Find \( P(E^c) \). \( 0.75 \)
18. Find \( P(F^c) \). \( 0.55 \)
19. If \( P(E) = 0.60 \), \( P(E \text{ or } F) = 0.85 \), and \( P(E \text{ and } F) = 0.05 \),
find \( P(F) \). \( 0.30 \)
20. If \( P(F) = 0.30 \), \( P(E \text{ or } F) = 0.65 \), and \( P(E \text{ and } F) = 0.15 \),
find \( P(E) \). \( 0.50 \)

In Problems 21–24, a golf ball is selected at random from a golf bag. If the golf bag contains 9 Titleists, 8 Maxflis, and 3 Top-Flites,
find the probability of each event.
21. The golf ball is a Titleist or Maxflie. \( \frac{17}{20} \)
22. The golf ball is a Maxflie or Top-Flite. \( \frac{11}{20} \)
23. The golf ball is not a Titleist. \( \frac{11}{20} \)
24. The golf ball is not a Top-Flite. \( \frac{17}{20} \)

Applying the Concepts

25. **Weapon of Choice** The following probability model shows

<table>
<thead>
<tr>
<th>Weapon</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gun</td>
<td>0.671</td>
</tr>
<tr>
<td>Knife</td>
<td>0.126</td>
</tr>
<tr>
<td>Blunt object</td>
<td>0.044</td>
</tr>
<tr>
<td>Personal weapon</td>
<td>0.065</td>
</tr>
<tr>
<td>Strangulation</td>
<td>0.010</td>
</tr>
<tr>
<td>Fire</td>
<td>0.009</td>
</tr>
<tr>
<td>Other</td>
<td>0.077</td>
</tr>
</tbody>
</table>

(a) Verify that this is a probability model.
(b) What is the probability that a randomly selected murder resulted from a gun or knife? Interpret this probability. \( 0.797 \)
(c) What is the probability that a randomly selected murder resulted from a knife, blunt object, or strangulation? Interpret this probability. \( 0.180 \)
(d) What is the probability that a randomly selected murder resulted from a weapon other than a gun? Interpret this probability. \( 0.329 \)
(e) Are murders by strangulation unusual? Yes

26. **Doctorates Conferred** The following probability model shows

<table>
<thead>
<tr>
<th>Area of Study</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>Engineering</td>
<td>0.148</td>
</tr>
<tr>
<td>Physical sciences</td>
<td>0.100</td>
</tr>
<tr>
<td>Life sciences</td>
<td>0.171</td>
</tr>
<tr>
<td>Mathematics</td>
<td>0.028</td>
</tr>
<tr>
<td>Computer sciences</td>
<td>0.026</td>
</tr>
<tr>
<td>Social sciences</td>
<td>0.172</td>
</tr>
<tr>
<td>Humanities</td>
<td>0.114</td>
</tr>
<tr>
<td>Education</td>
<td>0.144</td>
</tr>
<tr>
<td>Professional and other</td>
<td>0.097</td>
</tr>
</tbody>
</table>

(a) Verify that this is a probability model.
(b) What is the probability that a randomly selected doctoral candidate who earned a degree in 2005 studied physical science or life science? Interpret this probability. \( 0.271 \)
(c) What is the probability that a randomly selected doctoral candidate who earned a degree in 2005 studied physical science, life science, mathematics, or computer science? Interpret this probability. \( 0.325 \)
(d) What is the probability that a randomly selected doctoral candidate who earned a degree in 2005 did not study mathematics? Interpret this probability. \( 0.972 \)
(e) Are doctoral degrees in mathematics unusual? Does this result surprise you? Yes

27. If events \( E \) and \( F \) are disjoint and the events \( F \) and \( G \) are disjoint, must the events \( E \) and \( G \) necessarily be disjoint? Give an example to illustrate your opinion. No

28. Draw a Venn diagram like that in Figure 10 that expands the general addition rule to three events. Use the diagram to write the General Addition Rule for three events.

29. **Multiple Births** The following data represent the number of live multiple-delivery births (three or more babies) in 2005 for women 15 to 54 years old:

<table>
<thead>
<tr>
<th>Age</th>
<th>Number of Multiple Births</th>
</tr>
</thead>
<tbody>
<tr>
<td>15–19</td>
<td>83</td>
</tr>
<tr>
<td>20–24</td>
<td>465</td>
</tr>
<tr>
<td>25–29</td>
<td>1,635</td>
</tr>
<tr>
<td>30–34</td>
<td>2,443</td>
</tr>
<tr>
<td>35–39</td>
<td>1,604</td>
</tr>
<tr>
<td>40–44</td>
<td>344</td>
</tr>
<tr>
<td>45–54</td>
<td>120</td>
</tr>
</tbody>
</table>

Total 6,694

(a) Determine the probability that a randomly selected multiple birth in 2005 for women 15 to 54 years old involved a mother 30 to 39 years old. Interpret this probability. \( 0.605 \)
(b) Determine the probability that a randomly selected multiple birth in 2005 for women 15 to 54 years old involved a mother who was not 30 to 39 years old. Interpret this probability. \( 0.395 \)
(c) Determine the probability that a randomly selected multiple birth in 2005 for women 15 to 54 years old...
involved a mother who was less than 45 years old. Interpret this probability. \(0.982\)

(d) Determine the probability that a randomly selected multiple birth in 2005 for women 15 to 54 years old involved a mother who was at least 20 years old. Interpret this probability. \(0.968\)

30. **Housing** The following probability model shows the distribution for the number of rooms in U.S. housing units.

<table>
<thead>
<tr>
<th>Rooms</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>One</td>
<td>0.005</td>
</tr>
<tr>
<td>Two</td>
<td>0.011</td>
</tr>
<tr>
<td>Three</td>
<td>0.088</td>
</tr>
<tr>
<td>Four</td>
<td>0.183</td>
</tr>
<tr>
<td>Five</td>
<td>0.230</td>
</tr>
<tr>
<td>Six</td>
<td>0.204</td>
</tr>
<tr>
<td>Seven</td>
<td>0.123</td>
</tr>
<tr>
<td>Eight or more</td>
<td>0.156</td>
</tr>
</tbody>
</table>

Source: U.S. Census Bureau

(a) Verify that this is a probability model. 30. (b) \(0.896\)

(b) What is the probability that a randomly selected housing unit has four or more rooms? Interpret this probability.

(c) What is the probability that a randomly selected housing unit has fewer than eight rooms? Interpret this probability. \(0.844\)

(d) What is the probability that a randomly selected housing unit has from four to six (inclusive) rooms? Interpret this probability. \(0.617\)

(e) What is the probability that a randomly selected housing unit has at least two rooms? Interpret this probability. \(0.955\)

31. **A Deck of Cards** A standard deck of cards contains 52 cards, as shown in Figure 9. One card is randomly selected from the deck.

(a) Compute the probability of randomly selecting a heart or club from a deck of cards. \(1/2\)

(b) Compute the probability of randomly selecting a heart or club or diamond from a deck of cards. \(3/4\)

(c) Compute the probability of randomly selecting an ace or heart from a deck of cards. \(4/13\)

32. **A Deck of Cards** A standard deck of cards contains 52 cards, as shown in Figure 9. One card is randomly selected from the deck.

(a) Compute the probability of randomly selecting a two or three from a deck of cards. \(2/13\)

(b) Compute the probability of randomly selecting a two or three or four from a deck of cards. \(3/13\)

(c) Compute the probability of randomly selecting a two or club from a deck of cards. \(4/13\)

33. **Birthdays** Exclude leap years from the following calculations:

(a) Compute the probability that a randomly selected person does not have a birthday on November 8. \(364/365\)

(b) Compute the probability that a randomly selected person does not have a birthday on the 1st day of a month. \(359/365\)

(c) Compute the probability that a randomly selected person does not have a birthday on the 31st day of a month. \(354/365\)

(d) Compute the probability that a randomly selected person was not born in December. \(334/365\)

33. (b) \(353/365\) 33. (c) \(358/365\)

34. **Roulette** In the game of roulette, a wheel consists of 38 slots numbered 0, 00, 1, 2, \ldots, 36. The odd-numbered slots are red, and the even-numbered slots are black. The numbers 0 and 00 are green. To play the game, a metal ball is spun around the wheel and is allowed to fall into one of the numbered slots.

(a) What is the probability that the metal ball lands on green or red? \(10/19\)

(b) What is the probability that the metal ball does not land on green? \(18/19\)

35. **Health Problems** According to the Centers for Disease Control, the probability that a randomly selected citizen of the United States has hearing problems is 0.151. The probability that a randomly selected citizen of the United States has vision problems is 0.093. Can we compute the probability of randomly selecting a citizen of the United States who has hearing problems or vision problems by adding these probabilities? Why or why not? No

36. **Visits to the Doctor** A National Ambulatory Medical Care Survey administered by the Centers for Disease Control found that the probability a randomly selected patient visited the doctor for a blood pressure check is 0.593. The probability a randomly selected patient visited the doctor for urinalysis is 0.064. Can we compute the probability of randomly selecting a patient who visited the doctor for a blood pressure check or urinalysis by adding these probabilities? Why or why not? No

37. **Foster Care** A social worker for a child advocacy center has a caseload of 24 children under the age of 18. Her caseload by age is as follows:

<table>
<thead>
<tr>
<th>Age (yr)</th>
<th>Under 1</th>
<th>1–5</th>
<th>6–10</th>
<th>11–14</th>
<th>15–17</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cases</td>
<td>1</td>
<td>6</td>
<td>5</td>
<td>7</td>
<td>5</td>
</tr>
</tbody>
</table>

What is the probability that one of her clients, selected at random, is:

(a) Between 6 and 10 years old? Is this unusual? 0.208; no

(b) More than 5 years old? 0.708

(c) Less than 1 year old? Is this unusual? 0.042; yes

38. **Language Spoken at Home** According to the U.S. Census Bureau, the probability that a randomly selected household speaks only English at home is 0.81. The probability that a randomly selected household speaks only Spanish at home is 0.12.

(a) What is the probability that a randomly selected household speaks only English or only Spanish at home? 0.93

(b) What is the probability that a randomly selected household speaks a language other than only English or only Spanish at home? 0.07

(c) What is the probability that a randomly selected household speaks a language other than only English at home? 0.19

(d) Can the probability that a randomly selected household speaks only Polish at home equal 0.08? Why or why not? No

39. **Getting to Work** According to the U.S. Census Bureau, the probability that a randomly selected worker primarily drives a car to work is 0.867. The probability that a randomly selected worker primarily takes public transportation to work is 0.048.
Chapter 5 Probability

(a) What is the probability that a randomly selected worker primarily drives a car or takes public transportation to work? 0.895
(b) What is the probability that a randomly selected worker neither drives a car nor takes public transportation to work? 0.085
(c) What is the probability that a randomly selected worker does not drive a car to work? 0.133
(d) Can the probability that a randomly selected worker walks to work equal 0.15? Why or why not? No

40. Working Couples A guidance counselor at a middle school collected the following information regarding the employment status of married couples within his school’s boundaries.

<table>
<thead>
<tr>
<th>Number of Children under 18 Years Old</th>
<th>Worked</th>
<th>0</th>
<th>1</th>
<th>2 or More</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Husband only</td>
<td>172</td>
<td>79</td>
<td>174</td>
<td>425</td>
<td></td>
</tr>
<tr>
<td>Wife only</td>
<td>94</td>
<td>17</td>
<td>15</td>
<td>126</td>
<td></td>
</tr>
<tr>
<td>Both spouses</td>
<td>522</td>
<td>257</td>
<td>370</td>
<td>1,149</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>788</td>
<td>353</td>
<td>559</td>
<td>1,700</td>
<td></td>
</tr>
</tbody>
</table>

(a) What is the probability that, for married couple selected at random, both spouses work? 0.676
(b) What is the probability that, for married couple selected at random, the couple has one child under the age of 18? 0.208
(c) What is the probability that, for married couple selected at random, the couple has two or more children under the age of 18 and both spouses work? 0.218
(d) What is the probability that, for married couple selected at random, the couple has no children or only the husband works? 0.612
(e) Would it be unusual to select a married couple at random for which only the wife works? No

41. Cigar Smoking The data in the following table show the results of a national study of 137,243 U.S. men that investigated the association between cigar smoking and death from cancer. Note: Current cigar smoker means cigar smoker at time of death.

<table>
<thead>
<tr>
<th>Died from Cancer</th>
<th>Did Not Die from Cancer</th>
</tr>
</thead>
<tbody>
<tr>
<td>Never smoked cigars</td>
<td>782</td>
</tr>
<tr>
<td>Former cigar smoker</td>
<td>91</td>
</tr>
<tr>
<td>Current cigar smoker</td>
<td>141</td>
</tr>
</tbody>
</table>

Source: Shapiro, Jacobs, and Thun. “Cigar Smoking in Men and Risk of Death from Tobacco-Related Cancers,” Journal of the National Cancer Institute, February 16, 2000.

(a) If an individual is randomly selected from this study, what is the probability that he died from cancer? 0.007
(b) If an individual is randomly selected from this study, what is the probability that he was a current cigar smoker? 0.057
(c) If an individual is randomly selected from this study, what is the probability that he died from cancer and was a current cigar smoker? 0.001
(d) If an individual is randomly selected from this study, what is the probability that he died from cancer or was a current cigar smoker? 0.064

42. Civilian Labor Force The following table represents the employment status and gender of the civilian labor force ages 16 to 24 (in millions).

<table>
<thead>
<tr>
<th></th>
<th>Male</th>
<th>Female</th>
</tr>
</thead>
<tbody>
<tr>
<td>Employed</td>
<td>9.9</td>
<td>9.5</td>
</tr>
<tr>
<td>Unemployed</td>
<td>1.4</td>
<td>1.0</td>
</tr>
</tbody>
</table>


(a) What is the probability that a randomly selected 16- to 24-year-old individual from the civilian labor force is employed? 0.890
(b) What is the probability that a randomly selected 16- to 24-year-old individual from the civilian labor force is male? 0.518
(c) What is the probability that a randomly selected 16- to 24-year-old individual from the civilian labor force is employed and male? 0.454
(d) What is the probability that a randomly selected 16- to 24-year-old individual from the civilian labor force is employed or male? 0.854

43. Student Government Satisfaction Survey The Committee on Student Life at a university conducted a survey of 375 undergraduate students regarding satisfaction with student government. Results of the survey are shown in the table by class rank.

<table>
<thead>
<tr>
<th></th>
<th>Freshman</th>
<th>Sophomore</th>
<th>Junior</th>
<th>Senior</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Satisfied</td>
<td>57</td>
<td>49</td>
<td>64</td>
<td>61</td>
<td>231</td>
</tr>
<tr>
<td>Neutral</td>
<td>23</td>
<td>15</td>
<td>16</td>
<td>11</td>
<td>65</td>
</tr>
<tr>
<td>Not satisfied</td>
<td>21</td>
<td>18</td>
<td>14</td>
<td>26</td>
<td>79</td>
</tr>
<tr>
<td>Total</td>
<td>101</td>
<td>82</td>
<td>94</td>
<td>98</td>
<td>375</td>
</tr>
</tbody>
</table>

(a) If a study participant is selected at random, what is the probability that he or she is satisfied with student government? 0.616
(b) If a study participant is selected at random, what is the probability that he or she is a junior? 0.251
(c) If a study participant is selected at random, what is the probability that he or she is satisfied and is a junior? 0.171
(d) If a survey participant is selected at random, what is the probability that he or she is satisfied or is a junior? 0.696

44. The Placebo Effect A company is testing a new medicine for migraine headaches. In the study, 150 women were given the new medicine and an additional 100 women were given a placebo. Each participant was directed to take the medicine when the first symptoms of a migraine occurred and then to record whether the headache went away within 45 minutes or lingered. The results are recorded in the following table:

<table>
<thead>
<tr>
<th>Headache Went Away</th>
<th>Given medicine</th>
<th>Given placebo</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>132</td>
<td>18</td>
</tr>
</tbody>
</table>

(a) If a study participant is selected at random, what is the probability she was given the placebo? 0.40
(b) If a study participant is selected at random, what is the probability her headache went away within 45 minutes? 0.752
(c) If a study participant is selected at random, what is the probability she was given the placebo and her headache went away within 45 minutes? 0.224

(d) If a study participant is selected at random, what is the probability she was given the placebo or her headache went away within 45 minutes? 0.928

45. Active Duty The following table represents the number of active-duty military personnel by rank in the four major branches of the military as of December 31, 2007.

<table>
<thead>
<tr>
<th></th>
<th>Officer</th>
<th>Enlisted</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Army</td>
<td>84,781</td>
<td>428,929</td>
<td>513,710</td>
</tr>
<tr>
<td>Navy</td>
<td>51,167</td>
<td>278,193</td>
<td>329,360</td>
</tr>
<tr>
<td>Air Force</td>
<td>64,927</td>
<td>260,798</td>
<td>325,725</td>
</tr>
<tr>
<td>Marines</td>
<td>19,631</td>
<td>166,711</td>
<td>186,342</td>
</tr>
<tr>
<td>Total</td>
<td>220,506</td>
<td>1,134,631</td>
<td>1,355,137</td>
</tr>
</tbody>
</table>

Source: U.S. Department of Defense

45. (b) 0.243

(a) If an active-duty military person is selected at random, what is the probability the individual is an officer? 0.163
(b) If an active-duty military person is selected at random, what is the probability the individual is in the Navy? 0.038
(c) If an active-duty military person is selected at random, what is the probability the individual is a naval officer? 0.038
(d) If an active-duty military person is selected at random, what is the probability the individual is an officer or is in the Navy? 0.360

46. Driver Fatalities The following data represent the number of drivers in fatal crashes in the United States in 2005 by age group for male and female drivers:

<table>
<thead>
<tr>
<th>Age</th>
<th>Males</th>
<th>Females</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Under 16</td>
<td>227</td>
<td>77</td>
<td>304</td>
</tr>
<tr>
<td>16–20</td>
<td>5,180</td>
<td>2,113</td>
<td>7,293</td>
</tr>
<tr>
<td>21–34</td>
<td>13,611</td>
<td>4,311</td>
<td>17,922</td>
</tr>
<tr>
<td>35–54</td>
<td>15,108</td>
<td>5,027</td>
<td>20,135</td>
</tr>
<tr>
<td>55–74</td>
<td>6,801</td>
<td>2,452</td>
<td>9,253</td>
</tr>
<tr>
<td>Over 74</td>
<td>2,022</td>
<td>980</td>
<td>3,002</td>
</tr>
<tr>
<td>Total</td>
<td>42,949</td>
<td>14,960</td>
<td>57,909</td>
</tr>
</tbody>
</table>


(a) Determine the probability that a randomly selected driver involved in a fatal crash was male. 0.742
(b) Determine the probability that a randomly selected driver involved in a fatal crash was 16 to 20 years old. 0.126
(c) Determine the probability that a randomly selected driver involved in a fatal crash was a 16- to 20-year-old male. 0.089
(d) Determine the probability that a randomly selected driver involved in a fatal crash was male or 16 to 20 years old. 0.776
(e) Would it be unusual for a randomly selected driver involved in a fatal crash to be a 16- to 20-year-old female? Yes

47. Putting It Together: Red Light Cameras In a study of the feasibility of a red-light camera program in the city of Milwaukee, the following data were provided summarizing the projected number of crashes at 13 selected intersections over a 5-year period.

<table>
<thead>
<tr>
<th>Crash Type</th>
<th>Current System</th>
<th>With Red-Light Cameras</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reported injury</td>
<td>289</td>
<td>221</td>
</tr>
<tr>
<td>Reported property damage only</td>
<td>392</td>
<td>333</td>
</tr>
<tr>
<td>Unreported injury</td>
<td>78</td>
<td>60</td>
</tr>
<tr>
<td>Unreported property damage only</td>
<td>362</td>
<td>308</td>
</tr>
<tr>
<td>Total</td>
<td>1,121</td>
<td>922</td>
</tr>
</tbody>
</table>

Source: Krig, Moran, Regan. “An Analysis of a Red-Light Camera Program in the City of Milwaukee,” Spring 2006, prepared for the city of Milwaukee Budget and Management Division

47. (e) Current: 86.2; Camera: 70.9

(a) Identify the variables presented in the table.
(b) State whether each variable is qualitative or quantitative. If quantitative, state whether it is discrete or continuous.
(c) Construct a relative frequency distribution for each system.
(d) Construct a side-by-side relative frequency bar graph for the data.
(e) Compute the mean number of crashes per intersection in the study, if possible. If not possible, explain why.
(f) Compute the standard deviation number of crashes, if possible. If not possible, explain why.
(g) Based on the data shown, does it appear that the red-light camera program will be beneficial in reducing crashes at the intersections? Explain.
(h) For the current system, what is the probability that a crash selected at random will have reported injuries? 0.258
(i) For the camera system, what is the probability that a crash selected at random will have only property damage? 0.695

The study classified crashes further by indicating whether they were red-light running crashes or rear-end crashes. The results are as follows:

<table>
<thead>
<tr>
<th>Crash Type</th>
<th>Rear End Current</th>
<th>Rear End Cameras</th>
<th>Red-Light Running Current</th>
<th>Red-Light Running Cameras</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reported injury</td>
<td>67</td>
<td>77</td>
<td>222</td>
<td>144</td>
</tr>
<tr>
<td>Reported property damage only</td>
<td>157</td>
<td>180</td>
<td>235</td>
<td>153</td>
</tr>
<tr>
<td>Unreported injury</td>
<td>18</td>
<td>21</td>
<td>60</td>
<td>39</td>
</tr>
<tr>
<td>Unreported property damage only</td>
<td>145</td>
<td>167</td>
<td>217</td>
<td>141</td>
</tr>
<tr>
<td>Total</td>
<td>387</td>
<td>445</td>
<td>734</td>
<td>477</td>
</tr>
</tbody>
</table>

(j) Using Simpson’s Paradox, explain how the additional classification affects your response to part (g).
(k) What recommendation would you make to the city council regarding the implementation of the red-light camera program? Would you need any additional information before making your recommendation? Explain.
5.3 INDEPENDENCE AND THE MULTIPLICATION RULE

Objectives

1. Identify independent events
2. Use the Multiplication Rule for independent events
3. Compute at-least probabilities

1 Identify Independent Events

The Addition Rule for Disjoint Events deals with probabilities involving the word or. That is, it is used for computing the probability of observing an outcome in event \( E \) or event \( F \). We now describe a probability rule for computing the probability that \( E \) and \( F \) both occur.

Before we can present this rule, we must discuss the idea of independent events.

**Definition**

Two events \( E \) and \( F \) are independent if the occurrence of event \( E \) in a probability experiment does not affect the probability of event \( F \). Two events are dependent if the occurrence of event \( E \) in a probability experiment affects the probability of event \( F \).

To help you understand the idea of independence, we again look at a simple situation—flipping a coin. Suppose that you flip a fair coin twice. Does the fact that you obtained a head on the first toss have any effect on the likelihood of obtaining a head on the second toss? Not unless you are a master coin flipper who can manipulate the outcome of a coin flip! For this reason, the outcome from the first flip is independent of the outcome from the second flip. Let’s look at other examples.

**EXAMPLE 1**

Independent or Not?

(a) Suppose that you flip a coin and roll a die. The events “obtain a head” and “roll a 5” are independent because the results of the coin flip do not affect the results of the die toss.

(b) Are the events “earned a bachelor’s degree” and “earn more than $100,000 per year” independent? No, because knowing that an individual has a bachelor’s degree affects the likelihood that the individual is earning more than $100,000 per year.

(c) Two 24-year-old male drivers who live in the United States are randomly selected. The events “male 1 gets in a car accident during the year” and “male 2 gets in a car accident during the year” are independent because the males were randomly selected. This means what happens with one of the drivers has nothing to do with what happens to the other driver.

In Example 1(c), we are able to conclude that the events “male 1 gets in an accident” and “male 2 gets in an accident” are independent because the individuals are randomly selected. By randomly selecting the individuals, it is reasonable to conclude that the individuals are not related in any way (related in the sense that they do not live in the same town, attend the same school, and so on). If the two individuals did have a common link between them (such as they both lived on the same city block), then knowing that one male had a car accident may affect the likelihood that the other male had a car accident. After all, they could hit each other!

**Disjoint Events versus Independent Events**

It is important that we understand that disjoint events and independent events are different concepts. Recall that two events are disjoint if they have no outcomes in common. In other words, two events are disjoint if, knowing that one of the events
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Two events that are disjoint are not independent.

occurs, we know the other event did not occur. Independence means that one event occurring does not affect the probability of the other event occurring. Therefore, knowing two events are disjoint means that the events are not independent.

Consider the experiment of rolling a single die. Let $E$ represent the event “roll an even number,” and let $F$ represent the event “roll an odd number.” We can see that $E$ and $F$ are mutually exclusive because they have no outcomes in common. In addition, $P(E) = \frac{1}{2}$ and $P(F) = \frac{1}{2}$. However, if we are told that the roll of the die is going to be an even number, then what is the probability of event $F$? Because the outcome will be even, the probability of event $F$ is now 0 (and the probability of event $E$ is now 1).

2 Use the Multiplication Rule for Independent Events

Suppose that you flip a fair coin twice. What is the probability that you obtain a head on both flips? Put another way, what is the probability that you obtain a head on the first flip and you obtain a head on the second flip? We can easily create a sample space that lists the outcomes of this experiment. In flipping a coin twice, where $H$ represents the outcome “heads” and $T$ represents the outcome “tails,” we can obtain

$S = \{HH, HT, TH, TT\}$

There is one outcome with both heads. Because each outcome is equally likely, we have

$P(\text{heads on the 1st flip and heads on the 2nd flip}) = \frac{N(\text{heads on the 1st and heads on the 2nd})}{N(S)}$

$= \frac{1}{4}$

We may have intuitively been able to figure this out by recognizing $P(\text{head}) = \frac{1}{2}$ for each flip. So it seems reasonable that

$P(\text{heads on the 1st flip and heads on the 2nd flip}) = P(\text{heads on 1st flip}) \cdot P(\text{heads on 2nd flip})$

$= \frac{1}{2} \cdot \frac{1}{2}$

$= \frac{1}{4}$

Because both approaches result in the same answer, $\frac{1}{4}$, we conjecture that $P(E \text{ and } F) = P(E) \cdot P(F)$. Our conjecture is correct.

Multiplication Rule for Independent Events

If $E$ and $F$ are independent events, then

$P(E \text{ and } F) = P(E) \cdot P(F)$

EXAMPLE 2

Computing Probabilities of Independent Events

Problem: In the game of roulette, the wheel has slots numbered 0, 00, and 1 through 36. A metal ball is allowed to roll around a wheel until it falls into one of the numbered slots. You decide to play the game and place a bet on the number 17. What is the probability that the ball will land in the slot numbered 17 two times in a row?

Approach: There are 38 outcomes in the sample space of the experiment. We use the classical method of computing probabilities because the outcomes are equally
likely. In addition, we use the Multiplication Rule for Independent Events. The events “17 on first spin” and “17 on second spin” are independent because the ball does not remember it landed on 17 on the first spin, so this cannot affect the probability of landing on 17 on the second spin.

**Solution:** Because there are 38 possible outcomes to the experiment, the probability of the ball landing on 17 is \( \frac{1}{38} \). Because the events “17 on first spin” and “17 on second spin” are independent, we have

\[
P(\text{ball lands in slot 17 on the first spin and ball lands in slot 17 on the second spin}) = P(\text{ball lands in slot 17 on the first spin}) \cdot P(\text{ball lands in slot 17 on the second spin})
\]

\[
= \frac{1}{38} \cdot \frac{1}{38} = \frac{1}{1,444} \approx 0.0006925
\]

It is very unlikely that the ball will land on 17 twice in a row. We expect the ball to land on 17 twice in a row about 7 times in 10,000 trials.

We can extend the Multiplication Rule for three or more independent events.

**Multiplication Rule for \( n \) Independent Events**

If events \( E, F, G, \ldots \) are independent, then

\[
P(E \text{ and } F \text{ and } G \text{ and } \ldots) = P(E) \cdot P(F) \cdot P(G) \ldots
\]

---

**EXAMPLE 3**

**Life Expectancy**

**Problem:** The probability that a randomly selected 24 year old male will survive the year is 0.9986 according to the National Vital Statistics Report, Vol. 56, No. 9. What is the probability that 3 randomly selected 24-year-old males will survive the year? What is the probability that 20 randomly selected 24-year-old males will survive the year?

**Approach:** We can safely assume that the outcomes of the probability experiment are independent, because there is no indication that the survival of one male affects the survival of the others. For example, if two of the males lived in the same house, a house fire could kill both males and we lose independence. (Knowledge that one male died in a house fire certainly affects the probability that the other died.) By randomly selecting the males, we minimize the chances that they are related in any way.

**Solution**

\[
P(\text{all three males survive}) = P(1\text{st survives and 2nd survives and 3rd survives})
\]

\[
= P(1\text{st survives}) \cdot P(2\text{nd survives}) \cdot P(3\text{rd survives})
\]

\[
= (0.9986)(0.9986)(0.9986)
\]

\[
= 0.9958
\]

There is a 99.58% probability that all three males survive the year.

\[
P(\text{all 20 males survive}) = P(1\text{st survives and 2nd survives and \ldots and 20th survives})
\]

\[
= P(1\text{st survives}) \cdot P(2\text{nd survives}) \cdot \cdots \cdot P(20\text{th survives})
\]

\[
= (0.9986)(0.9986) \cdots (0.9986)
\]

\[
= (0.9986)^{20}
\]

\[
= 0.9724
\]

There is a 97.24% probability that all 20 males survive the year.
Compute At-Least Probabilities

We now present an example in which we compute at-least probabilities. These probabilities use the Complement Rule. The phrase at least means "greater than or equal to". For example, a person must be at least 17 years old to see an R-rated movie.

EXAMPLE 4

Computing At-Least Probabilities

Problem: Compute the probability that at least 1 male out of 1,000 aged 24 years will die during the course of the year if the probability that a randomly selected 24-year-old male survives the year is 0.9986.

Approach: The phrase at least means "greater than or equal to", so we wish to know the probability that 1 or 2 or 3 or ... or 1,000 males will die during the year. These events are mutually exclusive, so computing these probabilities is very time consuming. However, we notice that the complement of “at least one dying” is “none die.” We use the Complement Rule to compute the probability.

Solution

\[ P(\text{at least one dies}) = 1 - P(\text{none die}) \]
\[ = 1 - P(\text{1st survives and 2nd survives and ... and 1,000th survives}) \]
\[ = 1 - P(\text{1st survives}) \cdot P(\text{2nd survives}) \cdot \ldots \cdot P(\text{1,000th survives}) \quad \text{Independent events} \]
\[ = 1 - (0.9986)^{1,000} \]
\[ = 1 - 0.2464 \]
\[ = 0.7536 \]
\[ = 75.36\% \]

There is a 75.36% probability that at least one 24-year-old male out of 1,000 will die during the course of the year.

Summary: Rules of Probability

1. The probability of any event must be between 0 and 1, inclusive. If we let \( E \) denote any event, then \( 0 \leq P(E) \leq 1 \).
2. The sum of the probabilities of all outcomes must equal 1. That is, if the sample space \( S = \{e_1, e_2, \ldots, e_n\} \), then \( P(e_1) + P(e_2) + \cdots + P(e_n) = 1 \).
3. If \( E \) and \( F \) are disjoint events, then \( P(E \text{ or } F) = P(E) + P(F) \). If \( E \) and \( F \) are not disjoint events, then \( P(E \text{ or } F) = P(E) + P(F) - P(E \text{ and } F) \).
4. If \( E \) represents any event and \( E^c \) represents the complement of \( E \), then \( P(E^c) = 1 - P(E) \).
5. If \( E \) and \( F \) are independent events, then \( P(E \text{ and } F) = P(E) \cdot P(F) \).

Notice that or probabilities use the Addition Rule, whereas and probabilities use the Multiplication Rule. Accordingly, or probabilities imply addition, while and probabilities imply multiplication.
5.3 ASSESS YOUR UNDERSTANDING

**Concepts and Vocabulary**

1. Two events E and F are independent if the occurrence of event E in a probability experiment does not affect the probability of event F.
2. The word and in probability implies that we use the Multiplication Rule.
3. The word or in probability implies that we use the Addition Rule.
4. True or False: When two events are disjoint, they are also independent. False
5. If two events E and F are independent, \( P(E \text{ and } F) = P(E) \cdot P(F) \).
6. Suppose events E and F are disjoint. What is \( P(E \text{ and } F) \)? 0

**Skill Building**

7. Determine whether the events E and F are independent or dependent. Justify your answer.
   (a) E: Speeding on the interstate. F: Being pulled over by a police officer. Dependent
   (b) E: You gain weight. F: You eat fast food for dinner every night. Dependent
   (c) E: You get a high score on a statistics exam. F: The Boston Red Sox win a baseball game. Independent
8. Determine whether the events E and F are independent or dependent. Justify your answer.
   (a) E: The battery in your cell phone is dead. F: The batteries in your calculator are dead. Independent
   (b) E: Your favorite color is blue. F: Your friend’s favorite hobby is fishing. Independent
   (c) E: You are late for school. F: Your car runs out of gas. Dependent
9. Suppose that events E and F are independent, \( P(E) = 0.3 \), and \( P(F) = 0.6 \). What is the \( P(E \text{ and } F) \)? 0.18
10. Suppose that events E and F are independent, \( P(E) = 0.7 \), and \( P(F) = 0.9 \). What is the \( P(E \text{ and } F) \)? 0.63

**Applying the Concepts**

11. Flipping a Coin What is the probability of obtaining five heads in a row when flipping a coin? Interpret this probability. 0.03125
12. Rolling a Die What is the probability of obtaining 4 ones in a row when rolling a fair, six-sided die? Interpret this probability. 0.00008
13. Southpaws About 13% of the population is left-handed. If two people are randomly selected, what is the probability that both are left-handed? What is the probability that at least one is right-handed? 0.0169; 0.9831
14. Double Jackpot Shawn lives near the border of Illinois and Missouri. One weekend he decides to play $1 in both state lotteries in hopes of hitting two jackpots. The probability of winning the Missouri Lotto is about 0.00000028357 and the probability of winning the Illinois Lotto is about 0.000000098239.
   (a) Explain why the two lotteries are independent.
   (b) Find the probability that Shawn will win both jackpots. 0.0000000000000279

15. False Positives The ELISA is a test to determine whether the HIV antibody is present. The test is 99.5% effective. This means that the test will accurately come back negative if the HIV antibody is not present. The probability of a test coming back positive when the antibody is not present (a false positive) is 0.005. Suppose that the ELISA is given to five randomly selected people who do not have the HIV antibody.
   (a) What is the probability that the ELISA comes back negative for all five people? 0.9752
   (b) What is the probability that the ELISA comes back positive for at least one of the five people? 0.0248

16. Christmas Lights Christmas lights are often designed with a series circuit. This means that when one light burns out the entire string of lights goes black. Suppose that the lights are designed so that the probability a bulb will last 2 years is 0.995. The success or failure of a bulb is independent of the success or failure of other bulbs.
   (a) What is the probability that in a string of 100 lights all 100 will last 2 years? 0.6058
   (b) What is the probability that at least one bulb will burn out in 2 years? 0.3942

17. Life Expectancy The probability that a randomly selected 40-year-old male will live to be 41 years old is 0.99757, according to the National Vital Statistics Report, Vol. 56, No. 9.
   (a) What is the probability that two randomly selected 40-year-old males will live to be 41 years old? 0.99515
   (b) What is the probability that five randomly selected 40-year-old males will live to be 41 years old? 0.99791
   (c) What is the probability that at least one of five randomly selected 40-year-old males will not live to be 41 years old? Would it be unusual if at least one of five randomly selected 40-year-old males did not live to be 41 years old? 0.00209; yes

18. Life Expectancy The probability that a randomly selected 40-year-old female will live to be 41 years old is 0.99855 according to the National Vital Statistics Report, Vol. 56, No. 9.
   (a) What is the probability that two randomly selected 40-year-old females will live to be 41 years old? 0.99870
   (b) What is the probability that five randomly selected 40-year-old females will live to be 41 years old? 0.99827
   (c) What is the probability that at least one of five randomly selected 40-year-old females will not live to be 41 years old? Would it be unusual if at least one of five randomly selected 40-year-old females did not live to be 41 years old? 0.00723; yes

19. Blood Types Blood types can be classified as either Rh⁺ or Rh⁻. According to the Information Please Almanac, 99% of the Chinese population has Rh⁺ blood.
   (a) What is the probability that two randomly selected Chinese people have Rh⁺ blood? 0.9801
   (b) What is the probability that six randomly selected Chinese people have Rh⁺ blood? 0.9415
   (c) What is the probability that at least one of six randomly selected Chinese people has Rh⁻ blood? Would it be unusual that at least one of six randomly selected Chinese people has Rh⁻ blood? 0.0585; no

20. Quality Control Suppose that a company selects two people who work independently inspecting two-by-four timbers.
23. Cold Streaks

Players in sports are said to have “hot streaks” and “cold streaks.” For example, a batter in baseball might be considered to be in a slump, or cold streak, if he has made 10 outs in 10 consecutive at-bats. Suppose that at-bats are independent. Use this result to determine the probability that a hitter makes 10 outs in 10 consecutive at-bats, assuming that at-bats are independent events. Hint: The hitter makes an out 70% of the time.

(a) What is the probability that all 10 outs will be positive? 0.02835
(b) Are cold streaks unusual? Yes
(c) Interpret the probability from part (a).

24. Hot Streaks In a recent basketball game, a player who makes 65% of his free throws made eight consecutive free throws. Assuming that free-throw shots are independent, determine whether this feat was unusual.

(a) Would it be unusual to observe one component fail? Two components? No; yes
(b) What is the probability that a parallel structure with 2 identical components will succeed? 0.9775
(c) How many components would be needed in the structure so that the probability the system will succeed is greater than 0.9999? 5

25. Bowling Suppose that Ralph gets a strike when bowling 30% of the time.

(a) What is the probability that Ralph gets two strikes in a row? 0.09
(b) What is the probability that Ralph gets a turkey (three strikes in a row)? 0.027
(c) When events are independent, their complements are independent as well. Use this result to determine the probability that Ralph gets a strike and then does not get a strike. 0.21

26. NASCAR Fans Among Americans who consider themselves auto racing fans, 59% identify NASCAR stock cars as their favorite type of racing. Suppose that four auto racing fans are randomly selected.

(a) What is the probability that all four will identify NASCAR stock cars as their favorite type of racing? 0.1212
(b) What is the probability that at least one will not identify NASCAR stock cars as his or her favorite type of racing? 0.8788
(c) What is the probability that none will identify NASCAR stock cars as his or her favorite type of racing? 0.0283
(d) What is the probability that at least one will identify NASCAR stock cars as his or her favorite type of racing? 0.9717

27. Driving under the Influence Among 21- to 25-year-olds, 29% say they have driven while under the influence of alcohol. Suppose that three 21- to 25-year-olds are selected at random. Source: U.S. Department of Health and Human Services, reported in USA Today

(a) What is the probability that all three have driven while under the influence of alcohol? 0.0244
(b) What is the probability that at least one has not driven while under the influence of alcohol? 0.9756
(c) What is the probability that none of the three has driven while under the influence of alcohol? 0.3579
(d) What is the probability that at least one has driven while under the influence of alcohol? 0.6421

28. Defense System Suppose that a satellite defense system is established in which four satellites acting independently have a 0.9 probability of detecting an incoming ballistic missile. What is the probability that at least one of the four satellites detects an incoming ballistic missile? Would you feel safe with such a system?

(a) What is the probability that all four will succeed? 0.9775
(b) What is the probability that at least one will succeed? 0.9999
(c) How many components would be needed in the structure so that the probability the system will succeed is greater than 0.9999? 5

29. Audits For the fiscal year 2007, the IRS audited 1.77% of individual tax returns with income of $100,000 or more. Suppose this percentage stays the same for the current tax year.

(a) Would it be unusual for a return with income of $100,000 or more to be audited? Yes
(b) What is the probability that two randomly selected returns with income of $100,000 or more will be audited? 0.0351
(c) What is the probability that two randomly selected returns with income of $100,000 or more will not be audited? 0.9649
(d) What is the probability that at least one of two randomly selected returns with income of $100,000 or more will be audited? 0.0351

30. Casino Visits According to a December 2007 Gallup poll, 24% of American adults have visited a casino in the past 12 months.

(a) What is the probability that 4 randomly selected adult Americans have visited a casino in the past 12 months? 0.0033
(b) What is the probability that 4 randomly selected adult Americans have not visited a casino in the past 12 months? 0.9967
(c) What is the probability that 4 randomly selected adult Americans have visited a casino in the past 12 months, given that at least one has visited? 0.0351

31. Betting on Sports According to a Gallup Poll, about 17% of adult Americans bet on professional sports. Census data indicate that 48.4% of the adult population in the United States is male.

(a) Are the events “male” and “bet on professional sports” mutually exclusive? Explain. No
(b) Assuming that betting is independent of gender, compute the probability that an American adult selected at random is male and bets on professional sports. 0.0823
(c) Using the result in part (b), compute the probability that an American adult selected at random is male or bets on professional sports. 0.5717
Chapter 5  Probability

(d) The Gallup poll data indicated that 10.6% of adults in the United States are males and bet on professional sports. What does this indicate about the assumption in part (b)? Not independent.

(e) How will the information in part (d) affect the probability you computed in part (c)? 0.546

32. Fingerprints  Fingerprints are now widely accepted as a form of identification. In fact, many computers today use fingerprint identification to link the owner to the computer. In 1892, Sir Francis Galton explored the use of fingerprints to uniquely identify an individual. A fingerprint consists of ridgelines. Based on empirical evidence, Galton estimated the probability that a square consisting of six ridgelines that covered a fingerprint could be filled in accurately by an experienced fingerprint analyst as \( \frac{1}{2} \).

(a) Assuming that a full fingerprint consists of 24 of these squares, what is the probability that all 24 squares could be filled in correctly; assuming that success or failure in filling in one square is independent of success or failure in filling in any other square within the region? (This value represents the probability that two individuals would share the same ridgeline features within the 24-square region.)

(b) Galton further estimated that the likelihood of determining the fingerprint type (e.g. arch, left loop, whorl, etc.) as \( \left( \frac{1}{2} \right)^4 \) and the likelihood of the occurrence of the correct number of ridges entering and exiting each of the 24 regions as \( \left( \frac{1}{2} \right)^8 \). Assuming that all three probabilities are independent, compute Galton’s estimate of the probability that a particular fingerprint configuration would occur in nature (that is, the probability that a fingerprint match occurs by chance).

32. (a) \( \left( \frac{1}{2} \right)^{24} \approx 5.96 \times 10^{-8} \)  32. (b) \( \left( \frac{1}{2} \right)^{36} \approx 1.46 \times 10^{-11} \)

5.4  CONDITIONAL PROBABILITY AND THE GENERAL MULTIPLICATION RULE

Objectives

1. Compute conditional probabilities
2. Compute probabilities using the General Multiplication Rule

**Compute Conditional Probabilities**

In the last section, we learned that when two events are independent the occurrence of one event has no effect on the probability of the second event. However, we cannot generally assume that two events will be independent. Will the probability of being in a car accident change depending on driving conditions? We would expect so. For example, we would expect the probability of an accident to be higher for nighttime driving on icy roads than of daytime driving on dry roads. What about the likelihood of contracting a sexually transmitted disease (STD)? Do you think the number of sexual partners will affect the likelihood of contracting an STD?

According to data from the Centers for Disease Control, 33.3% of adult men in the United States are obese. So the probability is 0.333 that a randomly selected adult male in the United States is obese. However, 28% of adult men aged 20 to 39 are obese compared to 40% of adult men aged 40 to 59. The probability is 0.28 that an adult male is obese, given that they are aged 20 to 39. The probability is 0.40 that an adult male is obese, given that they are aged 40 to 59. The probability that an adult male is obese changes depending on the age group in which the individual falls. This is called conditional probability.

**Definition**

Conditional Probability

The notation \( P(F | E) \) is read “the probability of event \( F \) given event \( E \).” It is the probability that the event \( F \) occurs, given that the event \( E \) has occurred.

Let’s look at an example.
EXAMPLE 1

An Introduction to Conditional Probability

Problem: Suppose that a single die is rolled. What is the probability that the die comes up 3? Now suppose that the die is rolled a second time, but we are told the outcome will be an odd number. What is the probability that the die comes up 3?

Approach: We assume that the die is fair and compute the probabilities using equally likely outcomes.

Solution: In the first instance, there are six possibilities in the sample space, \( S = \{1, 2, 3, 4, 5, 6\} \), so \( P(3) = \frac{1}{6} \). In the second instance, there are three possibilities in the sample space, because the only possible outcomes are odd, so \( S = \{1, 3, 5\} \).

We express this probability symbolically as \( P(3|\text{outcome is odd}) = \frac{1}{3} \), which is read “the probability of rolling a 3, given that the outcome is odd, is one-third.”

So conditional probabilities reduce the size of the sample space under consideration. Let’s look at another example. The data in Table 8 represent the marital status of males and females 18 years old or older in the United States in 2006.

<table>
<thead>
<tr>
<th></th>
<th>Males (in millions)</th>
<th>Females (in millions)</th>
<th>Totals (in millions)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Never married</td>
<td>30.3</td>
<td>25.0</td>
<td>55.3</td>
</tr>
<tr>
<td>Married</td>
<td>63.6</td>
<td>64.1</td>
<td>127.7</td>
</tr>
<tr>
<td>Widowed</td>
<td>2.6</td>
<td>11.3</td>
<td>13.9</td>
</tr>
<tr>
<td>Divorced</td>
<td>9.7</td>
<td>13.1</td>
<td>22.8</td>
</tr>
<tr>
<td>Totals (in millions)</td>
<td>106.2</td>
<td>113.5</td>
<td>219.7</td>
</tr>
</tbody>
</table>

Source: U.S. Census Bureau, Current Population Reports

We want to know the probability that a randomly selected individual 18 years old or older is widowed. This probability is found by dividing the number of widowed individuals by the total number of individuals who are 18 years old or older.

\[
P(\text{widowed}) = \frac{13.9}{219.7} = 0.063
\]

Now suppose that we know the individual is female. Does this change the probability that she is widowed? Because the sample space now consists only of females, we can determine the probability that the individual is widowed, given that the individual is female, as follows:

\[
P(\text{widowed}|\text{female}) = \frac{N(\text{widowed females})}{N(\text{females})} = \frac{11.3}{113.5} = 0.100
\]

So, knowing that the individual is female increases the likelihood that the individual is widowed. This leads to the following.
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We used the second method for computing conditional probabilities in the widow example.

**Conditional Probability Rule**

If $E$ and $F$ are any two events, then

$$P(F|E) = \frac{P(E \text{ and } F)}{P(E)} = \frac{N(E \text{ and } F)}{N(E)} \quad (1)$$

The probability of event $F$ occurring, given the occurrence of event $E$, is found by dividing the probability of $E$ and $F$ by the probability of $E$. Or the probability of event $F$ occurring, given the occurrence of event $E$, is found by dividing the number of outcomes in $E$ and $F$ by the number of outcomes in $E$.

---

**EXAMPLE 2**

Conditional Probabilities on Marital Status and Gender

Problem: The data in Table 8 represent the marital status and gender of the residents of the United States aged 18 years old or older in 2006.

(a) Compute the probability that a randomly selected male has never married.

(b) Compute the probability that a randomly selected individual who has never married is male.

Approach

(a) We are given that the randomly selected person is male, so we concentrate on the male column. There are 106.2 million males and 30.3 million males who never married, so $N(\text{male}) = 106.2$ million and $N(\text{male and never married}) = 30.3$ million. Compute the probability using the Conditional Probability Rule.

(b) We are given that the randomly selected person has never married, so we concentrate on the never married row. There are 55.3 million people who have never married and 30.3 million males who have never married, so $N(\text{never married}) = 55.3$ million and $N(\text{male and never married}) = 30.3$ million. Compute the probability using the Conditional Probability Rule.

Solution

(a) Substituting into Formula (1), we obtain

$$P(\text{never married}|\text{male}) = \frac{N(\text{never married and male})}{N(\text{male})} = \frac{30.3}{106.2} = 0.285$$

There is a 28.5% probability that the randomly selected individual has never married, given that he is male.

(b) Substituting into Formula (1), we obtain

$$P(\text{male}|\text{never married}) = \frac{N(\text{male and never married})}{N(\text{never married})} = \frac{30.3}{55.3} = 0.548$$

There is a 54.8% probability that the randomly selected individual is male, given that he or she has never married.

---

What is the difference between the results of Example 2(a) and (b)? In Example 2(a), we found that 28.5% of males have never married, whereas in Example 2(b) we found that 54.8% of individuals who have never married are male. Do you see the difference?
EXAMPLE 3

Birth Weights of Preterm Babies

Problem: In 2005, 12.64% of all births were preterm. (The gestation period of the pregnancy was less than 37 weeks.) Also in 2005, 0.22% of all births resulted in a preterm baby that weighed 8 pounds, 13 ounces or more. What is the probability that a randomly selected baby weighs 8 pounds, 13 ounces or more, given that the baby was preterm? Is this unusual?

Approach: We want to know the probability that the baby weighs 8 pounds, 13 ounces or more, given that the baby was preterm. We know that 0.22% of all babies weighed 8 pounds, 13 ounces or more and were preterm, so \( P(\text{weighs 8 pounds, 13 ounces or more and preterm}) = 0.22\% \). We also know that 12.64% of all births were preterm, so \( P(\text{preterm}) = 12.64\% \). We compute the probability by dividing the probability that a baby will weigh 8 pounds, 13 ounces or more and be preterm by the probability that a baby will be preterm.

Solution: \( P(\text{weighs 8 pounds, 13 ounces or more | preterm}) = \frac{P(\text{weighs 8 pounds, 13 ounces or more and preterm})}{P(\text{preterm})} = \frac{0.22\%}{12.64\%} = \frac{0.0022}{0.1264} \approx 0.0174 = 1.74\% \)

There is a 1.74% probability that a randomly selected baby will weigh 8 pounds, 13 ounces or more, given that the baby is preterm. It is unusual for preterm babies to weigh 8 pounds, 13 ounces or more.

Compute Probabilities Using the General Multiplication Rule

If we solve the Conditional Probability Rule for \( P(E \text{ and } F) \), we obtain the General Multiplication Rule.

**General Multiplication Rule**

The probability that two events \( E \) and \( F \) both occur is

\[
P(E \text{ and } F) = \frac{P(E)}{P(F | E)}
\]

In words, the probability of \( E \) and \( F \) is the probability of event \( E \) occurring times the probability of event \( F \) occurring, given the occurrence of event \( E \).

EXAMPLE 4

Using the General Multiplication Rule

Problem: The probability that a driver who is speeding gets pulled over is 0.8. The probability that a driver gets a ticket, given that he or she is pulled over, is 0.9. What is the probability that a randomly selected driver who is speeding gets pulled over and gets a ticket?

Approach: Let \( E \) represent the event “driver who is speeding gets pulled over,” and let \( F \) represent the event “driver gets a ticket.” We use the General Multiplication Rule to compute \( P(E \text{ and } F) \).

Solution: \( P(\text{driver who is speeding gets pulled over and gets a ticket}) = P(E \text{ and } F) = P(E) \cdot P(F | E) = 0.8(0.9) = 0.72 \). There is a 72% probability that a driver who is speeding gets pulled over and gets a ticket.
Acceptance Sampling

Problem: Suppose that a box of 100 circuits is sent to a manufacturing plant. Of the 100 circuits shipped, 5 are defective. The plant manager receiving the circuits randomly selects 2 and tests them. If both circuits work, she will accept the shipment. Otherwise, the shipment is rejected. What is the probability that the plant manager discovers at least 1 defective circuit and rejects the shipment?

Approach: We wish to determine the probability that at least one of the tested circuits is defective. There are four possibilities in this probability experiment: Neither of the circuits is defective, the first is defective while the second is not, the first is not defective while the second is defective, or both circuits are defective. We cannot compute this probability using the fact that there are four outcomes and three that result in at least 1 defective, because the outcomes are not equally likely. We need a different approach. One approach is to use a tree diagram to list all possible outcomes and the General Multiplication Rule to compute the probability for each outcome. We could then determine the probability of at least 1 defective by adding the probability that the first is defective while the second is not, the first is not defective while the second is defective, or both are defective, using the Addition Rule (because they are disjoint).

A second approach is to compute the probability that both circuits are not defective and use the Complement Rule to determine the probability of at least 1 defective. We will illustrate both approaches.

Solution: We have 100 circuits and 5 of them are defective, so 95 circuits are not defective. To use our first approach, we construct a tree diagram to determine the possible outcomes for the experiment. We draw two branches corresponding to the two possible outcomes (defective or not defective) for the first repetition of the experiment (the first circuit). For the second circuit, we draw four branches: two branches originate from the first defective circuit and two branches originate from the first nondefective circuit. See Figure 12, where D stands for defective and G stands for good (not defective). Since the outcomes are not equally likely, we include the probabilities in our diagram to show how the probability of each outcome is obtained. The probability for each outcome is obtained by multiplying the individual probabilities along the corresponding path is the diagram.

Figure 12

<table>
<thead>
<tr>
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<tbody>
<tr>
<td></td>
<td>$\frac{94}{99}$</td>
<td>$\frac{95}{100}$</td>
<td>$\frac{5}{99}$</td>
<td>$\frac{95}{99}$</td>
<td>$\frac{5}{100}$</td>
<td>$\frac{4}{99}$</td>
</tr>
</tbody>
</table>

From our tree diagram, and using the Addition Rule, we can write

\[
P(\text{at least 1 defective}) = P(GD) + P(DG) + P(DD)
\]

\[
= 0.048 + 0.048 + 0.002
\]

\[
= 0.098
\]

There is a 9.8% probability that the shipment will not be accepted.
Section 5.4 Conditional Probability and the General Multiplication Rule

For our second approach, we compute the probability that both circuits are not defective and use the Complement Rule to determine the probability of at least 1 defective.

\[ P(\text{at least 1 defective}) = 1 - P(\text{none defective}) \]
\[ = 1 - P(1\text{st not defective}) \cdot P(2\text{nd not defective} \mid 1\text{st not defective}) \]
\[ = 1 - \left( \frac{95}{100} \right) \cdot \left( \frac{94}{99} \right) \]
\[ = 1 - 0.902 \]
\[ = 0.098 \]

There is a 9.8% probability that the shipment will not be accepted. This result is the same as that obtained using our tree diagram.

Whenever a small random sample is taken from a large population, it is reasonable to compute probabilities of events assuming independence. Consider the following example.

**EXAMPLE 6**

**Sickle-Cell Anemia**

**Problem:** A survey of 10,000 African Americans found that 27 had sickle-cell anemia.

(a) Suppose we randomly select 1 of the 10,000 African Americans surveyed. What is the probability that he or she will have sickle-cell anemia?

(b) If two individuals from this group are randomly selected, what is the probability that both have sickle-cell anemia?

(c) Compute the probability of randomly selecting two individuals from this group who have sickle-cell anemia, assuming independence.

**Approach:** We let the event \( E \) of African Americans who have sickle-cell anemia divided by the number in the survey. To answer part (b), we let \( E_1 = \text{“first person has sickle-cell anemia”} \) and \( E_2 = \text{“second person has sickle-cell anemia,”} \) and then we compute \( P(E_1 \text{ and } E_2) = P(E_1) \cdot P(E_2 \mid E_1) \). To answer part (c), we use the Multiplication Rule for Independent Events.

**Solution**

(a) If one individual is selected, \( P(E) = \frac{27}{10,000} = 0.0027 \).

(b) Using the Multiplication Rule, we have

\[ P(E_1 \text{ and } E_2) = P(E_1) \cdot P(E_2 \mid E_1) = \frac{27}{10,000} \cdot \frac{26}{9,999} \approx 0.00000702 \]

Notice that \( P(E_2 \mid E_1) = \frac{26}{9,999} \) because we are sampling without replacement, so after event \( E_1 \) occurs there is one less person with sickle-cell anemia and one less person in the sample space.

(c) The assumption of independence means that the outcome of the first trial of the experiment does not affect the probability of the second trial. (It is like sampling with replacement.) Therefore, we assume that

\[ P(E_1) = P(E_2) = \frac{27}{10,000} \]

Then

\[ P(E_1 \text{ and } E_2) = P(E_1) \cdot P(E_2) = \frac{27}{10,000} \cdot \frac{27}{10,000} \approx 0.00000729 \]
Two events are independent if the occurrence of event $E$ in a probability experiment does not affect the probability of event $F$. We can now express independence using conditional probabilities.

### Definition

Two events $E$ and $F$ are independent if $P(E|F) = P(E)$ or, equivalently, if $P(F|E) = P(F)$.

If either condition in our definition is true, the other is as well. In addition, for independent events,

$$P(E \text{ and } F) = P(E) \cdot P(F).$$

So the Multiplication Rule for Independent Events is a special case of the General Multiplication Rule.

Look back at Table 8 on page 293. Because $P(\text{widowed}) = 0.063$ does not equal $P(\text{widowed|female}) = 0.100$, the events “widowed” and “female” are not independent. In fact, knowing an individual is female increases the likelihood that the individual is also widowed.

### 5.4 ASSESS YOUR UNDERSTANDING

**Concepts and Vocabulary**

1. The notation $P(F|E)$ means the probability of event $F$ given event $E$.

2. If $P(E) = 0.6$ and $P(F|E) = 0.34$, are events $E$ and $F$ independent? \(\text{No}\)

**Skill Building**

3. Suppose that $E$ and $F$ are two events and that $P(E \text{ and } F) = 0.6$ and $P(E) = 0.8$. What is $P(F|E)$? \(0.75\)

4. Suppose that $E$ and $F$ are two events and that $P(E \text{ and } F) = 0.21$ and $P(E) = 0.4$. What is $P(F|E)$? \(0.525\)

5. Suppose that $E$ and $F$ are two events and that $N(E \text{ and } F) = 420$ and $N(E) = 740$. What is $P(F|E)$? \(0.566\)

6. Suppose that $E$ and $F$ are two events and that $N(E \text{ and } F) = 380$ and $N(E) = 925$. What is $P(F|E)$? \(0.411\)

7. Suppose that $E$ and $F$ are two events and that $P(E) = 0.8$ and $P(F|E) = 0.4$. What is $P(E \text{ and } F)$? \(0.32\)

8. Suppose that $E$ and $F$ are two events and that $P(E) = 0.4$ and $P(F|E) = 0.6$. What is $P(E \text{ and } F)$? \(0.24\)

9. According to the U.S. Census Bureau, the probability that a randomly selected individual in the United States earns more than $75,000 per year is 9.5%. The probability that a randomly selected individual in the United States earns more than $75,000 per year, given that the individual has earned a bachelor’s degree, is 21.5%. Are the events “earn more than $75,000 per year” and “earned a bachelor’s degree” independent? \(\text{No}\)

10. The probability that a randomly selected individual in the United States 25 years and older has at least a bachelor’s degree is 0.287. The probability that an individual in the United States 25 years and older has at least a bachelor’s degree, given that the individual is Hispanic, is 0.127. Are the events “bachelor’s degree” and “Hispanic” independent? \(\text{No}\)


### Applying the Concepts

11. **Drawing a Card** Suppose that a single card is selected from a standard 52-card deck. What is the probability that the card drawn is a club? Now suppose that a single card is drawn from a standard 52-card deck, but we are told that the card is black. What is the probability that the card drawn is a club?
17. (b) | 0.184

12. **Drawing a Card** Suppose that a single card is selected from a standard 52-card deck. What is the probability that the card drawn is a king? Now suppose that a single card is drawn from a standard 52-card deck, but we are told that the card is a heart. What is the probability that the card drawn is a king? Did the knowledge that the card is a heart change the probability that the card was a king? What term is used to describe this result? [1/13; 1/13; no independence]

13. **Rainy Days** For the month of June in the city of Chicago, 37% of the days are cloudy. Also in the month of June in the city of Chicago, 21% of the days are cloudy and rainy. What is the probability that a randomly selected day in June will be rainy if it is cloudy? | 0.568

14. **Cause of Death** According to the U.S. National Center for Health Statistics, in 2005, 0.15% of deaths in the United States were 25- to 34-year-olds whose cause of death was cancer. In addition, 1.71% of all those who died were 25 to 34 years old. What is the probability that a randomly selected death is the result of cancer if the individual is known to have been 25 to 34 years old? | 0.090

15. **High School Dropouts** According to the U.S. Census Bureau, 8.4% of high school dropouts are 16- to 17-year-olds. In addition, 6.2% of high school dropouts are white 16- to 17-year-olds. What is the probability that a randomly selected dropout is white, given that he or she is 16 to 17 years old? | 0.758

16. **Income by Region** According to the U.S. Census Bureau, 19.1% of U.S. households are in the Northeast. In addition, 4.4% of U.S. households earn $75,000 per year or more and are located in the Northeast. Determine the probability that a randomly selected U.S. household earns more than $75,000 per year, given that the household is located in the Northeast. | 0.488

17. **Health Insurance Coverage** The following data represent, in thousands, the type of health insurance coverage of people by age in the year 2006. | 16. 0.250

<table>
<thead>
<tr>
<th>Age</th>
<th>Private health insurance</th>
<th>Government health insurance</th>
<th>No health insurance</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>&lt;18</td>
<td>47,906</td>
<td>22,109</td>
<td>8,661</td>
<td>78,676</td>
</tr>
<tr>
<td>18–44</td>
<td>74,375</td>
<td>12,375</td>
<td>27,054</td>
<td>114,304</td>
</tr>
<tr>
<td>45–64</td>
<td>57,505</td>
<td>11,304</td>
<td>11,824</td>
<td>79,546</td>
</tr>
<tr>
<td>&gt;64</td>
<td>21,904</td>
<td>33,982</td>
<td>541</td>
<td>56,427</td>
</tr>
<tr>
<td>Total</td>
<td>201,690</td>
<td>80,270</td>
<td>46,993</td>
<td>328,953</td>
</tr>
</tbody>
</table>

Source: U.S. Census Bureau

18. **Cigar Smoking** The data in the following table show the results of a national study of 137,243 U.S. men that investigated the association between cigar smoking and death from cancer. **Note:** “Current cigar smoker” means cigar smoker at time of death.

<table>
<thead>
<tr>
<th>Cigar Smoking Status</th>
<th>Died from Cancer</th>
<th>Did Not Die from Cancer</th>
</tr>
</thead>
<tbody>
<tr>
<td>Never smoked cigars</td>
<td>782</td>
<td>120,747</td>
</tr>
<tr>
<td>Former cigar smoker</td>
<td>91</td>
<td>7,757</td>
</tr>
<tr>
<td>Current cigar smoker</td>
<td>141</td>
<td>7,725</td>
</tr>
</tbody>
</table>

Source: Shapiro, Jacobs, and Thun. “Cigar Smoking in Men and Risk of Death from Tobacco-Related Cancers,” *Journal of the National Cancer Institute*, February 16, 2000

(a) What is the probability that a randomly selected individual from the study who died from cancer was a former cigar smoker? | 0.090

(b) What is the probability that a randomly selected individual from the study who was a former cigar smoker died from cancer? | 0.012

19. **Traffic Fatalities** The following data represent the number of traffic fatalities in the United States in 2005 by person type for male and female drivers.

<table>
<thead>
<tr>
<th>Person Type</th>
<th>Male</th>
<th>Female</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Driver</td>
<td>20,795</td>
<td>6,598</td>
<td>27,393</td>
</tr>
<tr>
<td>Passenger</td>
<td>5,190</td>
<td>4,896</td>
<td>10,086</td>
</tr>
<tr>
<td>Total</td>
<td>25,985</td>
<td>11,494</td>
<td>37,479</td>
</tr>
</tbody>
</table>


(a) What is the probability that a randomly selected traffic fatality who was female was a passenger? | 0.426

(b) What is the probability that a randomly selected passenger fatality was female? | 0.485

(c) Suppose you are a police officer called to the scene of a traffic accident with a fatality. The dispatcher states that the victim was driving, but the gender is not known. Is the victim more likely to be male or female? Why? | Male

20. **Marital Status** The following data, in thousands, represent the marital status of Americans 25 years old or older and their levels of education in 2006.

<table>
<thead>
<tr>
<th>Did Not Graduate from High School</th>
<th>High School Graduate</th>
<th>Some College</th>
<th>College Graduate</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Never married</td>
<td>4,803</td>
<td>9,575</td>
<td>5,593</td>
<td>11,632</td>
</tr>
<tr>
<td>Married, spouse present</td>
<td>13,880</td>
<td>35,627</td>
<td>19,201</td>
<td>46,965</td>
</tr>
<tr>
<td>Married, spouse absent</td>
<td>1,049</td>
<td>1,049</td>
<td>503</td>
<td>944</td>
</tr>
<tr>
<td>Separated</td>
<td>1,162</td>
<td>1,636</td>
<td>790</td>
<td>1,060</td>
</tr>
<tr>
<td>Widowed</td>
<td>4,070</td>
<td>5,278</td>
<td>1,819</td>
<td>2,725</td>
</tr>
<tr>
<td>Divorced</td>
<td>2,927</td>
<td>7,725</td>
<td>4,639</td>
<td>7,323</td>
</tr>
<tr>
<td>Total</td>
<td>27,891</td>
<td>60,890</td>
<td>32,545</td>
<td>70,558</td>
</tr>
</tbody>
</table>


(a) What is the probability that a randomly selected individual who has never married is a high school graduate? | 0.303

(b) What is the probability that a randomly selected individual who is a high school graduate has never married? | 0.157
21. **Acceptance Sampling** Suppose that you just received a shipment of six televisions. Two of the televisions are defective. If two televisions are randomly selected, compute the probability that both televisions work. What is the probability that at least one does not work? 0.4; 0.6

22. **Committee** A committee consists of four women and three men. The committee will randomly select two people to attend a conference in Hawaii. Find the probability that both are women. \( \frac{2}{7} \)

23. Suppose that two cards are randomly selected from a standard 52-card deck.
   (a) What is the probability that the first card is a king and the second card is a king if the sampling is done without replacement?
   (b) What is the probability that the first card is a king and the second card is a king if the sampling is done with replacement?
   (c) Suppose that two cards are randomly selected from a standard 52-card deck. The probability that the first card is a club and the second card is a club if the sampling is done without replacement is \( \frac{1}{221} \).
   (d) What is the probability that the first card is a club and the second card is a club if the sampling is done with replacement?
   (e) What is the probability that the first card is a king and the second card is a king if the sampling is done without replacement? \( \frac{1}{169} \)

24. Suppose that two cards are randomly selected from a standard 52-card deck.
   (a) What is the probability that the first card is a club and the second card is a club if the sampling is done without replacement?
   (b) What is the probability that the first card is a club and the second card is a club if the sampling is done with replacement?
   (c) What is the probability that the first card is a club and the second card is a club if the sampling is done without replacement? \( \frac{1}{16} \)

25. **Board Work** This past semester, I had a small business calculus section. The students in the class were Mike, Neta, Jinita, Kristin, and Dave. Suppose that I randomly select two people to go to the board to work problems. What is the probability that Dave is the first person chosen to go to the board and Neta is the second? \( \frac{1}{20} \)

26. **Party** My wife has organized a monthly neighborhood party. Five people are involved in the group: Yolanda (my wife), Lorrie, Laura, Kim, and Anne Marie. They decide to randomly select the first and second home that will host the party. What is the probability that my wife hosts the first party and Lorrie hosts the second? Note: Once a home has hosted, it cannot host again until all other homes have hosted. \( \frac{1}{20} \)

27. **Playing a CD on the Random Setting** Suppose that a compact disk (CD) you just purchased has 13 tracks. After listening to the CD, you decide that you like 5 of the songs. With the random feature on your CD player, each of the 13 songs is played once in random order. Find the probability that among the first two songs played
   (a) You like both of them. Would this be unusual? \( \frac{5}{39} \); no
   (b) You like neither of them.
   (c) You like exactly one of them.
   (d) Redo (a)–(c) if a song can be replayed before all 13 songs are played (if, for example, track 2 can play twice in a row).
   (e) Determine the probability that both contain diet soda.
   (f) Determine the probability that both contain regular soda. Would this be unusual? \( \frac{6}{11} \)

28. **Packaging Error** Due to a manufacturing error, three cans of regular soda were accidentally filled with diet soda and placed into a 12-pack. Suppose that two cans are randomly selected from the 12-pack.
   (a) Compute the probability that 10 people have 10 different birthdays.
   Hint: The first person’s birthday can occur 365 ways, the second person’s birthday can occur 364 ways, because he or she cannot have the same birthday as the first person, the third person’s birthday can occur 363 ways, because
Section 5.4 Conditional Probability and the General Multiplication Rule

34. **The Birthday Problem** Using the procedure given in Problem 33, compute the probability that at least 2 people in a room of 23 people share the same birthday. 0.007

35. **Teen Communication** The following data represent the number of different communication activities (e.g., cell phone, text messaging, and e-mail) used by a random sample of teenagers in a given week.

<table>
<thead>
<tr>
<th>Activities</th>
<th>0</th>
<th>1–2</th>
<th>3–4</th>
<th>5–7</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Male</td>
<td>21</td>
<td>81</td>
<td>60</td>
<td>48</td>
<td>200</td>
</tr>
<tr>
<td>Female</td>
<td>21</td>
<td>52</td>
<td>56</td>
<td>71</td>
<td>200</td>
</tr>
<tr>
<td>Total</td>
<td>42</td>
<td>133</td>
<td>116</td>
<td>109</td>
<td>400</td>
</tr>
</tbody>
</table>

(a) Are the events “male” and “0 activities” independent? Justify your answer. Yes
(b) Are the events “female” and “5–7 activities” independent? Justify your answer. No
(c) Are the events “1–2 activities” and “3–4 activities” mutually exclusive? Justify your answer. Yes
(d) Are the events “male” and “1–2 activities” mutually exclusive? Justify your answer. No

36. **2008 Democratic Primary** The following data represent political party by age from an entrance poll during the Iowa caucus.

<table>
<thead>
<tr>
<th></th>
<th>17–29</th>
<th>30–44</th>
<th>45–64</th>
<th>65+</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Republican</td>
<td>224</td>
<td>340</td>
<td>1075</td>
<td>561</td>
<td>2,200</td>
</tr>
<tr>
<td>Democrat</td>
<td>184</td>
<td>384</td>
<td>773</td>
<td>459</td>
<td>1,800</td>
</tr>
<tr>
<td>Total</td>
<td>408</td>
<td>724</td>
<td>1,848</td>
<td>1,020</td>
<td>4,000</td>
</tr>
</tbody>
</table>

(a) Are the events “Republican” and “30–44” independent? Justify your answer. No
(b) Are the events “Democrat” and “65+” independent? Justify your answer. Yes
(c) Are the events “17–29” and “45–64” mutually exclusive? Justify your answer. Yes
(d) Are the events “Republican” and “45–64” mutually exclusive? Justify your answer. No

37. **A Flush** A flush in the card game of poker occurs if a player gets five cards that are all the same suit (clubs, diamonds, hearts, or spades). Answer the following questions to obtain the probability of being dealt a flush in five cards.

(a) We initially concentrate on one suit, say clubs. There are 13 clubs in a deck. Compute 
P(first card is clubs and second card is clubs and third card is clubs and fourth card is clubs and fifth card is clubs). 0.0000495
(b) A flush can occur if we get five clubs or five diamonds or five hearts or five spades. Compute 
P(five clubs or five diamonds or five hearts or five spades). Note that the events are mutually exclusive. 0.002

38. **A Royal Flush** A royal flush in the game of poker occurs if the player gets the cards Ten, Jack, Queen, King, and Ace all in the same suit. Use the procedure given in Problem 37 to compute the probability of being dealt a royal flush. 0.00000154

39. **Independence in Small Samples from Large Populations** Suppose that a computer chip company has just shipped 10,000 computer chips to a computer company. Unfortunately, 50 of the chips are defective.

(a) Compute the probability that two randomly selected chips are defective using conditional probability. 0.0000000237
(b) There are 50 defective chips out of 10,000 shipped. The probability that the first chip randomly selected is defective is 50/10,000 = 0.005 = 0.5%. Compute the probability that two randomly selected chips are defective under the assumption of independent events. Compare your results to part (a). Conclude that, when small samples are taken from large populations without replacement, the assumption of independence does not significantly affect the probability. 0.0000025

40. **Independence in Small Samples from Large Populations** Suppose that a poll is being conducted in the village of Lemont. The pollster identifies her target population as all residents of Lemont 18 years old or older. This population has 6,494 people.

(a) Compute the probability that the first resident selected to participate in the poll is Roger Cummings and the second is Rick Whittingham. 0.00000000237
(b) The probability that any particular resident of Lemont is the first person picked is 1/6,494. Compute the probability that Roger is selected first and Rick is selected second, assuming independence. Compare your results to part (a). Conclude that, when small samples are taken from large populations without replacement, the assumption of independence does not significantly affect the probability.

41. **Independent?** Refer to the contingency table in Problem 17 that relates age and health insurance coverage. Determine 
P(<18 years old) and 
P(<18 years old|no health insurance).

(a) Are the events “<18 years old” and “no health insurance” independent? 0.239; 0.184; no

42. **Independent?** Refer to the contingency table in Problem 18 that relates cigar smoking and deaths from cancer. Determine 
P(died from cancer) and 
P(died from cancer|current cigar smoker).

(a) Are the events “died from cancer” and “current cigar smoker” independent? 0.007; 0.018; no

43. **Independent?** Refer to the contingency table in Problem 19 that relates person type in a traffic fatality to gender. Determine 
P(female) and 
P(female|driver).

(a) Are the events “female” and “driver” independent? 0.307; 0.241; no

44. **Independent?** Refer to the contingency table in Problem 20 that relates marital status and level of education. Determine 
P(divorced) and 
P(divorced|college graduate).

(a) Are the events “divorced” and “college graduate” independent?

45. **Let’s Make a Deal** In 1991, columnist Marilyn vos Savant posted her reply to a reader’s question. The question posed was in reference to one of the games played on the gameshow Let’s Make a Deal hosted by Monty Hall.

Suppose you’re on a game show, and you’re given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what’s behind the doors, opens another door, say No. 3, which has a goat. He then says to you, “Do you want to pick door No. 2?” Is it to your advantage to take the switch?

(b) 0.0000000237

44. 0.117; 0.102; no
Her reply generated a tremendous amount of backlash, with many highly educated individuals angrily responding that she was clearly mistaken in her reasoning.

(a) Using subjective probability, estimate the probability of winning if you switch.

(b) Load the Let’s Make a Deal applet. Simulate the probability that you will win if you switch by going through the simulation at least 100 times. How does your simulated result compare to your answer to part (a)?

(c) Research the Monty Hall Problem as well as the reply by Marilyn Vos Savant. How does the probability she gives compare to the two estimates you obtained?

(d) Write a report detailing why Marilyn was correct. One approach is to use a random variable on a wheel similar to the one shown. On the wheel, the innermost ring indicates the door where the car is located, the middle ring indicates the door you selected, and the outer ring indicates the door(s) that Monty could show you. In the outer ring, green indicates you lose if you switch while purple indicates you win if you switch.

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**Consumer Reports**

**His ‘N’ Hers Razor?**

With so many men’s and women’s versions of different products, you might wonder how different they really are. To help answer this question, technicians at Consumers Union compared a new triple-edge razor for women with a leading double-edge razor for women and a leading triple-edge razor for men. The technicians asked 30 women panelists to shave with the razors over a 4-week period, following a random statistical design.

After each shave, the panelists were asked to answer a series of questions related to the performance of the razor. One question involved rating the razor. The following table contains a summary of the results for this question.

<table>
<thead>
<tr>
<th>Razor</th>
<th>Rating</th>
<th>Poor</th>
<th>Fair</th>
<th>Excellent</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td></td>
<td>1</td>
<td>8</td>
<td>21</td>
</tr>
<tr>
<td>B</td>
<td></td>
<td>0</td>
<td>11</td>
<td>19</td>
</tr>
<tr>
<td>C</td>
<td></td>
<td>6</td>
<td>11</td>
<td>13</td>
</tr>
</tbody>
</table>

Using the information in the table, answer the following questions:

(a) Calculate the probability that a randomly selected razor scored Excellent.

(b) Calculate the probability that a randomly selected razor scored Poor.

(c) Calculate the probability of randomly selecting Razor B, given the score was Fair.

(d) Calculate the probability of receiving an Excellent rating, given that Razor C was selected.

(e) Are razor type and rating independent?

(f) Which razor would you choose based on the information given? Support your decision.

**Note to Readers:** In many cases, our test protocol and analytical methods are more complicated than described in these examples. The data and discussions have been modified to make the material more appropriate for the audience.

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**5.5 COUNTING TECHNIQUES**

**Objectives**

1. Solve counting problems using the Multiplication Rule
2. Solve counting problems using permutations
3. Solve counting problems using combinations
4. Solve counting problems involving permutations with nondistinct items
5. Compute probabilities involving permutations and combinations

**Solve Counting Problems Using the Multiplication Rule**

Counting plays a major role in many diverse areas, including probability. In this section, we look at special types of counting problems and develop general techniques for solving them.

We begin with an example that demonstrates a general counting principle.
**EXAMPLE 1  Counting the Number of Possible Meals**

**Problem:** The fixed-price dinner at Mabenka Restaurant provides the following choices:

- **Appetizer:** soup or salad
- **Entrée:** baked chicken, broiled beef patty, baby beef liver, or roast beef au jus
- **Dessert:** ice cream or cheese cake

How many different meals can be ordered?

**Approach:** Ordering such a meal requires three separate decisions:

<table>
<thead>
<tr>
<th>Choose an Appetizer</th>
<th>Choose an Entrée</th>
<th>Choose a Dessert</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 choices</td>
<td>4 choices</td>
<td>2 choices</td>
</tr>
</tbody>
</table>

Figure 13 is a tree diagram that lists the possible meals that can be ordered.
Chapter 5 Probability

Solution: Look at the tree diagram in Figure 13. For each choice of appetizer, we have 4 choices of entrée, and that, for each of these 2 \cdot 4 = 8 choices, there are 2 choices for dessert. A total of

\[ 2 \cdot 4 \cdot 2 = 16 \]

different meals can be ordered.

Example 1 illustrates a general counting principle.

**Multiplication Rule of Counting**

If a task consists of a sequence of choices in which there are \( p \) selections for the first choice, \( q \) selections for the second choice, \( r \) selections for the third choice, and so on, then the task of making these selections can be done in

\[ p \cdot q \cdot r \cdots \]

different ways.

**EXAMPLE 2 Counting Airport Codes (Repetition Allowed)**

**Problem:** The International Airline Transportation Association (IATA) assigns three-letter codes to represent airport locations. For example, the code for Fort Lauderdale International Airport is FLL. How many different airport codes are possible?

**Approach:** We are choosing 3 letters from 26 letters and arranging them in order. We notice that repetition of letters is allowed. We use the Multiplication Rule of Counting, recognizing we have 26 ways to choose the first letter, 26 ways to choose the second letter, and 26 ways to choose the third letter.

**Solution:** By the Multiplication Rule,

\[ 26 \cdot 26 \cdot 26 = 26^3 = 17,576 \]

different airport codes are possible.

In Example 2, we were allowed to repeat letters. In the following example, repetition is not allowed.

**EXAMPLE 3 Counting without Repetition**

**Problem:** Three members from a 14-member committee are to be randomly selected to serve as chair, vice-chair, and secretary. The first person selected is the chair; the second is the vice-chair; and the third is the secretary. How many different committee structures are possible?

**Approach:** The task consists of making three selections. The first selection requires choosing from 14 members. Because a member cannot serve in more than one capacity, the second selection requires choosing from the 13 remaining members. The third selection requires choosing from the 12 remaining members. (Do you see why?) We use the Multiplication Rule to determine the number of possible committees.

**Solution:** By the Multiplication Rule,

\[ 14 \cdot 13 \cdot 12 = 2,184 \]

different committee structures are possible.

Now Work Problem 31
The Factorial Symbol

We now introduce a special symbol that can assist us in representing certain types of counting problems.

If \( n \geq 0 \) is an integer, the **factorial symbol**, \( n! \), is defined as follows:

\[
0! = 1, \quad 1! = 1, \quad n! = n(n - 1) \cdots 3 \cdot 2 \cdot 1
\]

For example, \( 2! = 2 \cdot 1 = 2 \), \( 3! = 3 \cdot 2 \cdot 1 = 6 \), \( 4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24 \), and so on. Table 9 lists the values of \( n! \) for \( 0 \leq n \leq 6 \).

Using Technology

Your calculator has a factorial key. Use it to see how fast factorials increase in value. Find the value of \( 69! \). What happens when you try to find \( 70! \)? In fact, \( 70! \) is larger than \( a \text{ googol} \) (a googol), the largest number most calculators can display.

Table 9

<table>
<thead>
<tr>
<th>( n )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n! )</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>6</td>
<td>24</td>
<td>120</td>
<td>720</td>
</tr>
</tbody>
</table>

The Traveling Salesperson

**Problem:** You have just been hired as a book representative for Pearson Education. On your first day, you must travel to seven schools to introduce yourself. How many different routes are possible?

**Approach:** The seven schools are different. Let’s call the schools \( A, B, C, D, E, F, \) and \( G \). School \( A \) can be visited first, second, third, fourth, fifth, sixth, or seventh. So, we have seven choices for school \( A \). We would then have six choices for school \( B \), five choices for school \( C \), and so on. We can use the Multiplication Rule and the factorial to find our solution.

**Solution:** \( 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 7! = 5,040 \) different routes are possible.

Now Work Problems 5 and 33

---

Solve Counting Problems Using Permutations

Examples 3 and 4 illustrate a type of counting problem referred to as a *permutation*.

A permutation is an ordered arrangement in which \( r \) objects are chosen from \( n \) distinct (different) objects and repetition is not allowed. The symbol \( _nP_r \) represents the number of permutations of \( r \) objects selected from \( n \) objects.

So we could represent the solution to the question posed in Example 3 as

\[
_7P_3 = 7 \cdot 6 \cdot 5 = 210
\]

and the solution to Example 4 could be represented as

\[
_7P_7 = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5,040
\]

To arrive at a formula for \( _nP_r \), we note that there are \( n \) choices for the first selection, \( n - 1 \) choices for the second selection, \( n - 2 \) choices for the third selection, \( \ldots \), and \( n - (r - 1) \) choices for the \( r \)th selection. By the Multiplication Rule, we have

\[
_nP_r = n \cdot (n - 1) \cdot (n - 2) \cdots [n - (r - 1)]
\]

\[
= n \cdot (n - 1) \cdot (n - 2) \cdots (n - r + 1)
\]
This formula for \( nP_r \) can be written in factorial notation:
\[
nP_r = n \cdot (n - 1) \cdot (n - 2) \cdot \cdots \cdot (n - r + 1)
\]
\[
= n \cdot (n - 1) \cdot (n - 2) \cdot \cdots \cdot (n - r + 1) \cdot \frac{(n - r) \cdots 3 \cdot 2 \cdot 1}{(n - r) \cdots 3 \cdot 2 \cdot 1}
\]
\[
= \frac{n!}{(n - r)!}
\]

We have the following result.

**Number of Permutations of \( n \) Distinct Objects Taken \( r \) at a Time**

The number of arrangements of \( r \) objects chosen from \( n \) objects, in which
1. the \( n \) objects are distinct,
2. repetition of objects is not allowed, and
3. order is important,
is given by the formula
\[
nP_r = \frac{n!}{(n - r)!} \tag{1}
\]

**EXAMPLE 5**

**Computing Permutations**

**Problem:** Evaluate:
(a) \( 7P_5 \) \hspace{1cm} (b) \( 8P_2 \) \hspace{1cm} (c) \( 5P_5 \)

**Approach:** To answer (a), we use Formula (1) with \( n = 7 \) and \( r = 5 \). To answer (b), we use Formula (1) with \( n = 8 \) and \( r = 2 \). To answer (c), we use Formula (1) with \( n = 5 \) and \( r = 5 \).

**Solution**

(a) \( 7P_5 = \frac{7!}{(7 - 5)!} = \frac{7!}{2!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2!}{2!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3}{2 \cdot 1} = \frac{2520}{2} = 1260 \)\( \frac{5 \text{ factors}}{} \)

(b) \( 8P_2 = \frac{8!}{(8 - 2)!} = \frac{8!}{6!} = \frac{8 \cdot 7 \cdot 6!}{6!} = \frac{8 \cdot 7}{1} = 56 \)\( \frac{2 \text{ factors}}{} \)

(c) \( 5P_5 = \frac{5!}{(5 - 5)!} = \frac{5!}{0!} = 5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120 \)

Example 5(c) illustrates a general result:
\[
P_r = r!
\]

**EXAMPLE 6**

**Computing Permutations Using Technology**

**Problem:** Evaluate \( 7P_5 \) using statistical software or a graphing calculator with advanced statistical features.

**Approach:** We will use both Excel and a TI-84 Plus graphing calculator to evaluate \( 7P_5 \). The steps for computing permutations using Excel and the TI-83/84 Plus graphing calculator can be found in the Technology Step-by-Step on page 315.
Section 5.5 Counting Techniques

Solution: Figure 14(a) shows the result in Excel, and Figure 14(b) shows the result on a TI-84 Plus graphing calculator.

**Figure 14**

Here is the result. If you click OK, the result goes into the cell.

(a) ![Excel screenshot](image)

(b) ![TI-84 Plus graphing calculator screenshot](image)

**EXAMPLE 7**

**Betting on the Trifecta**

**Problem:** In how many ways can horses in a 10-horse race finish first, second, and third?

**Approach:** The 10 horses are distinct. Once a horse crosses the finish line, that horse will not cross the finish line again, and, in a race, order is important. We have a permutation of 10 objects taken 3 at a time.

**Solution:** The top three horses can finish a 10-horse race in

\[ \text{10P}_3 = \frac{10!}{(10 - 3)!} = \frac{10!}{7!} = \frac{10 \cdot 9 \cdot 8 \cdot 7!}{7!} = \frac{10 \cdot 9 \cdot 8}{3 \text{ factors}} = 720 \text{ ways} \]

Now Work Problem 45

**3 Solve Counting Problems Using Combinations**

In a permutation, order is important. For example, the arrangements \(ABC, ACB, BAC, BCA, CAB, \) and \(CBA\) are considered different arrangements of the letters \(A, B, \) and \(C.\) If order is unimportant, the six arrangements of the letters \(A, B, \) and \(C\) given above are not different. That is, we do not distinguish \(ABC\) from \(BAC.\) In the card game of poker, the order in which the cards are received does not matter. The combination of the cards is what matters.

**Definition**

A *combination* is a collection, without regard to order, of \(n\) distinct objects without repetition. The symbol \(\binom{n}{r}\) represents the number of combinations of \(n\) distinct objects taken \(r\) at a time.
Listing Combinations

Problem: Roger, Rick, Randy, and Jay are going to play golf. They will randomly select teams of two players each. List all possible team combinations. That is, list all the combinations of the four people Roger, Rick, Randy, and Jay taken two at a time. What is \( 4C_2 \)?

Approach: We list the possible teams. We note that order is unimportant, so \{Roger, Rick\} is the same as \{Rick, Roger\}.

Solution: The list of all such teams (combinations) is

Roger, Rick; Roger, Randy; Roger, Jay; Rick, Randy; Rick, Jay; Randy, Jay

So,

\[ 4C_2 = 6 \]

There are six ways of forming teams of two from a group of four players.

We can find a formula for \( nC_r \) by noting that the only difference between a permutation and a combination is that we disregard order in combinations. To determine \( nC_r \), we eliminate from the formula for \( nP_r \) the number of permutations that were rearrangements of a given set of \( r \) objects. In Example 8, for example, selecting \{Roger, Rick\} was the same as selecting \{Rick, Roger\}, so there were \( 2! = 2 \) rearrangements of the two objects. This can be determined from the formula for \( nP_r \) by calculating \( nP_r = r! \). So, if we divide \( nP_r \) by \( r! \), we will have the desired formula for \( nC_r \):

\[
\frac{nC_r}{r!} = \frac{n!}{r!(n-r)!}
\]

We have the following result.

Number of Combinations of \( n \) Distinct Objects Taken \( r \) at a Time

The number of different arrangements of \( n \) objects using \( r \leq n \) of them, in which

1. the \( n \) objects are distinct,
2. repetition of objects is not allowed, and
3. order is not important

is given by the formula

\[
\frac{n!}{r!(n-r)!}
\]

Using Formula (2) to solve the problem presented in Example 8, we obtain

\[ 4C_2 = \frac{4!}{2!(4-2)!} = \frac{4!}{2!2!} = \frac{4 \cdot 3 \cdot 2!}{2 \cdot 1 \cdot 2!} = \frac{12}{2} = 6 \]

Using Formula (2)

Problem: Use Formula (2) to find the value of each expression.

(a) \( 3C_1 \)  
(b) \( 5C_4 \)  
(c) \( 6C_2 \)

Approach: We use Formula (2): \( nC_r = \frac{n!}{r!(n-r)!} \)
**Solution**

(a) \( \binom{4}{2} = \frac{4!}{2!(4-2)!} = \frac{4!}{2!2!} = 6 \quad n = 4, r = 1 \)

(b) \( \binom{6}{4} = \frac{6!}{4!(6-4)!} = \frac{6!}{4!2!} = 15 \quad n = 6, r = 4 \)

(c) \( \binom{6}{2} = \frac{6!}{2!(6-2)!} = \frac{6!}{2!4!} = 15 \quad n = 6, r = 2 \)

Notice in Example 9 that \( \binom{n}{r} = \binom{n}{n-r} \). This result can be generalized:

\[ nC_r = nC_{n-r} \]

**EXAMPLE 10**

**Computing Combinations Using Technology**

**Problem:** Evaluate \( \binom{n}{r} \) using statistical software or a graphing calculator with advanced statistical features.

**Approach:** We will use both Excel and a TI-84 Plus graphing calculator to evaluate \( \binom{n}{r} \). The steps for computing combinations using Excel and the TI-83/84 Plus graphing calculator can be found in the Technology Step-by-Step on page 315.

**Solution:** Figure 15(a) shows the result in Excel, and Figure 15(b) shows the result on a TI-84 Plus graphing calculator.

**EXAMPLE 11**

**Simple Random Samples**

**Problem:** How many different simple random samples of size 4 can be obtained from a population whose size is 20?

**Approach:** The 20 individuals in the population are distinct. In addition, the order in which individuals are selected is unimportant. Thus, the number of simple random samples of size 4 from a population of size 20 is a combination of 20 objects taken 4 at a time.

**Solution:** Use Formula (2) with \( n = 20 \) and \( r = 4 \):

\[ \binom{20}{4} = \frac{20!}{4!(20-4)!} = \frac{20!}{4!16!} = \frac{20 \cdot 19 \cdot 18 \cdot 17 \cdot 16!}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 16!} = \frac{116,280}{24} = 4,845 \]

There are 4,845 different simple random samples of size 4 from a population whose size is 20.
### Solve Counting Problems Involving Permutations with Nondistinct Items

Sometimes we wish to arrange objects in order, but some of the objects are not distinguishable.

#### DNA Structure

**Problem:** The DNA structure is a double helix, similar to a ladder that is twisted in a spiral shape. Each rung of the ladder consists of a pair of DNA bases. A DNA sequence consists of a series of letters representing a DNA strand that spells out the genetic code. A DNA nucleotide contains a molecule of sugar, a molecule of phosphoric acid, and a base molecule. There are four possible letters (A, C, G, and T), each representing a specific nucleotide base in the DNA strand (adenine, cytosine, guanine, and thymine, respectively). How many distinguishable sequences can be formed using two As, two Cs, three Gs, and one T? **Note:** Scientists estimate that the complete human genome contains roughly 3 billion nucleotide base pairs.

**Approach:** Each sequence formed will have eight letters. To construct each sequence, we need to fill in eight positions with the eight letters:

\[
\begin{array}{cccccccc}
T & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\end{array}
\]

The process of forming a sequence consists of four tasks:

- **Task 1:** Choose the positions for the two As.
- **Task 2:** Choose the positions for the two Cs.
- **Task 3:** Choose the positions for the three Gs.
- **Task 4:** Choose the position for the one T.

Task 1 can be done in \( \binom{8}{2} \) ways. This leaves 6 positions to be filled, so task 2 can be done in \( \binom{6}{2} \) ways. This leaves 4 positions to be filled, so task 3 can be done in \( \binom{4}{3} \) ways. The last position can be filled in \( \binom{1}{1} \) way.

**Solution:** By the Multiplication Rule, the number of possible sequences that can be formed is

\[
\binom{8}{2} \cdot \binom{6}{2} \cdot \binom{4}{3} \cdot \binom{1}{1} = \frac{8!}{2! \cdot 6!} \cdot \frac{6!}{2! \cdot 4!} \cdot \frac{4!}{3! \cdot 1!} \cdot \frac{1!}{1!} = \frac{8!}{2! \cdot 2! \cdot 3! \cdot 1! \cdot 0!} = 1,680
\]

There are 1,680 possible distinguishable sequences that can be formed.

The solution to Example 12 is suggestive of a general result. Had the letters in the sequence each been different, \( 8! \) possible sequences would have been formed. This is the numerator of the answer. The presence of two As, two Cs, and three Gs reduces the number of different sequences, as the entries in the denominator illustrate. We are led to the following result:

#### Permutations with Nondistinct Items

The number of permutations of \( n \) objects of which \( n_1 \) are of one kind, \( n_2 \) are of a second kind, \ldots, and \( n_k \) are of a \( k \)th kind is given by

\[
\frac{n!}{n_1! \cdot n_2! \cdots n_k!}
\]

where \( n = n_1 + n_2 + \cdots + n_k \).
**EXAMPLE 13**

**Arranging Flags**

**Problem:** How many different vertical arrangements are there of 10 flags if 5 are white, 3 are blue, and 2 are red?

**Approach:** We seek the number of permutations of 10 objects, of which 5 are of one kind (white), 3 are of a second kind (blue), and 2 are of a third kind (red).

**Solution:** Using Formula (3), we find that there are

\[
\frac{10!}{5! \cdot 3! \cdot 2!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5!}{5! \cdot 3! \cdot 2!} = 2,520
\]

different vertical arrangements.

**Summary**

To summarize the differences between combinations and the various types of permutations, we present Table 10.

<table>
<thead>
<tr>
<th>Description</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>Combination</td>
<td>( \binom{n}{r} = \frac{n!}{r!(n-r)!} )</td>
</tr>
<tr>
<td>Permutation of Distinct Items with Replacement</td>
<td>( n^r )</td>
</tr>
<tr>
<td>Permutation of Distinct Items without Replacement</td>
<td>( n^P_r = \frac{n!}{(n-r)!} )</td>
</tr>
<tr>
<td>Permutation of Nondistinct Items without Replacement</td>
<td>( \frac{n!}{n_1!n_2!\cdots n_k!} )</td>
</tr>
</tbody>
</table>

**Compute Probabilities Involving Permutations and Combinations**

The counting techniques presented in this section can be used to determine probabilities of certain events by using the classical method of computing probabilities. Recall that this method stated the probability of an event \( E \) is the number of ways event \( E \) can occur divided by the number of different possible outcomes of the experiment provided each outcome is equally likely.

**EXAMPLE 14**

**Winning the Lottery**

**Problem:** In the Illinois Lottery, an urn contains balls numbered 1 to 52. From this urn, six balls are randomly chosen without replacement. For a $1 bet, a player chooses two sets of six numbers. To win, all six numbers must match those chosen from the urn. The order in which the balls are selected does not matter. What is the probability of winning the lottery?

**Approach:** The probability of winning is given by the number of ways a ticket could win divided by the size of the sample space. Each ticket has two sets of six numbers, so there are two chances (for the two sets of numbers) of winning for each ticket. The size of the sample space \( S \) is the number of ways that 6 objects can be
Chapter 5 Probability

selected from 52 objects without replacement and without regard to order, so

\[ N(S) = \binom{52}{6}. \]

**Solution:** The size of the sample space is

\[ N(S) = \binom{52}{6} = \frac{52!}{6!(52 - 6)!} = \frac{52 \cdot 51 \cdot 49 \cdot 48 \cdot 47 \cdot 46!}{6! \cdot 46!} = 20,358,520 \]

Each ticket has two sets of 6 numbers, so a player has two chances of winning for each $1. If \( E \) is the event “winning ticket,” then \( N(E) = 2 \). The probability of \( E \) is

\[ P(E) = \frac{2}{20,358,520} = 0.000000098 \]

There is about a 1 in 10,000,000 chance of winning the Illinois Lottery!

**EXAMPLE 15**

**Probabilities Involving Combinations**

**Problem:** A shipment of 120 fasteners that contains 4 defective fasteners was sent to a manufacturing plant. The quality-control manager at the manufacturing plant randomly selects 5 fasteners and inspects them. What is the probability that exactly 1 fastener is defective?

**Approach:** The probability that exactly 1 fastener is defective is found by calculating the number of ways of selecting exactly 1 defective fastener in 5 fasteners and dividing this result by the number of ways of selecting 5 fasteners from 120 fasteners. To choose exactly 1 defective in the 5 requires choosing 1 defective from the 4 defectives and 4 nondefectives from the 116 nondefectives. The order in which the fasteners are selected does not matter, so we use combinations.

**Solution:** The number of ways of choosing 1 defective fastener from 4 defective fasteners is \( \binom{4}{1} \). The number of ways of choosing 4 nondefective fasteners from 116 nondefectives is \( \binom{116}{4} \). Using the Multiplication Rule, we find that the number of ways of choosing 1 defective and 4 nondefective fasteners is

\[ (\binom{4}{1}) \cdot (\binom{116}{4}) = 4 \cdot 7,160,245 = 28,640,980 \]

The number of ways of selecting 5 fasteners from 120 fasteners is \( \binom{120}{5} = 190,578,024 \). The probability of selecting exactly 1 defective fastener is

\[ P(1 \text{ defective fastener}) = \frac{(\binom{4}{1})(\binom{116}{4})}{\binom{120}{5}} = \frac{28,640,980}{190,578,024} = 0.1503 \]

There is a 15.03% probability of randomly selecting exactly 1 defective fastener.

**Concepts and Vocabulary**

1. A **permutation** is an ordered arrangement of \( r \) objects chosen from \( n \) distinct objects without repetition.
2. A **combination** is an arrangement of \( r \) objects chosen from \( n \) distinct objects without repetition and without regard to order.
3. **True or False:** In a combination problem, order is not important.  **True**
4. Explain the difference between a combination and a permutation.

**Skill Building**

In Problems 5–10, find the value of each factorial.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Answer</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.</td>
<td>5!</td>
</tr>
<tr>
<td>6.</td>
<td>7!</td>
</tr>
<tr>
<td>7.</td>
<td>10!</td>
</tr>
<tr>
<td>8.</td>
<td>12!</td>
</tr>
<tr>
<td>9.</td>
<td>0!</td>
</tr>
<tr>
<td>10.</td>
<td>1!</td>
</tr>
</tbody>
</table>

In Problems 11–18, find the value of each permutation.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Answer</th>
</tr>
</thead>
<tbody>
<tr>
<td>11.</td>
<td>( _6P_2 )</td>
</tr>
<tr>
<td>12.</td>
<td>( _3P_2 )</td>
</tr>
<tr>
<td>13.</td>
<td>( _4P_4 )</td>
</tr>
<tr>
<td>14.</td>
<td>( _7P_7 )</td>
</tr>
</tbody>
</table>
15. $aP_0 = 1$
16. $aP_0 = 1$
17. $aP_3 = 336$
18. $aP_4 = 3,024$

In Problems 19–26, find the value of each combination.
19. $aC_3 = 56$
20. $aC_2 = 36$
21. $aC_2 = 45$
22. $aC_3 = 220$
23. $aC_1 = 52$
24. $aC_4 = 1$
25. $aC_3 = 17,296$
26. $aC_4 = 27,405$

27. List all the permutations of five objects $a$, $b$, $c$, $d$, and $e$ taken two at a time without repetition. What is $aP_2$? 20
28. List all the permutations of four objects $a$, $b$, $c$, and $d$ taken two at a time without repetition. What is $aP_2$? 12
29. List all the combinations of five objects $a$, $b$, $c$, $d$, and $e$ taken two at a time. What is $aC_2$? 10
30. List all the combinations of four objects $a$, $b$, $c$, and $d$ taken two at a time. What is $aC_2$? 6

Applying the Concepts

31. Clothing Options A man has six shirts and four ties. Assuming that they all match, how many different shirt-and-tie combinations can he wear? 24
32. Clothing Options A woman has five blouses and three skirts. Assuming that they all match, how many different outfits can she wear? 15
33. Arranging Songs on a CD Suppose Dan is going to burn a compact disk (CD) that will contain 12 songs. In how many ways can Dan arrange the 12 songs on the CD? 12!
34. Arranging Students In how many ways can 15 students be lined up? 15!
35. Traveling Salesperson A salesperson must travel to eight cities to promote a new marketing campaign. How many different trips are possible if any route between cities is possible? 8!
36. Randomly Playing Songs A certain compact disk player randomly plays each of 10 songs on a CD. Once a song is played, it is not repeated until all the songs on the CD have been played. In how many different ways can the CD player play the 10 songs? 10!
37. Stocks on the NYSE Companies whose stocks are listed on the New York Stock Exchange (NYSE) have their company name represented by either one, two, or three letters (repetition of letters is allowed). What is the maximum number of companies that can be listed on the New York Stock Exchange? 18,278
38. Stocks on the NASDAQ Companies whose stocks are listed on the NASDAQ stock exchange have their company name represented by either four or five letters (repetition of letters is allowed). What is the maximum number of companies that can be listed on the NASDAQ? 12,338,352
39. Garage Door Code Outside a home there is a keypad that will open the garage if the correct four-digit code is entered.

Section 5.5 Counting Techniques

44. 2,600,000
(a) How many codes are possible? 10,000
(b) What is the probability of entering the correct code on the first try, assuming that the owner doesn’t remember the code? 0.0001

40. Social Security Numbers A Social Security number is used to identify each resident of the United States uniquely. The number is of the form xxx–xx–xxxx, where each $x$ is a digit from 0 to 9. 40. (b) 1/10^9
(a) How many Social Security numbers can be formed? 10^9
(b) What is the probability of correctly guessing the Social Security number of the president of the United States?

41. User Names Suppose that a local area network requires eight letters for user names. Lower- and uppercase letters are considered the same. How many user names are possible for the local area network? 26^8

42. User Names How many additional user names are possible in Problem 41 if the last character could also be a digit? 10 x 26^7

43. Combination Locks A combination lock has 50 numbers on it. To open it, you turn counterclockwise to a number, then rotate clockwise to a second number, and then counterclockwise to the third number. Repetitions are allowed.
(a) How many different lock combinations are there? 50^3
(b) What is the probability of guessing a lock combination on the first try? 1/150^3

44. Forming License Plate Numbers How many different license plate numbers can be made by using one letter followed by five digits selected from the digits 0 through 9?

45. INDY 500 Suppose 40 cars start at the Indianapolis 500. In how many ways can the top 3 cars finish the race? 59,280

46. Betting on the Perfecta In how many ways can the top 2 horses finish in a 10-horse race? 90

47. Forming a Committee Four members from a 20-person committee are to be selected randomly to serve as chairperson, vice-chairperson, secretary, and treasurer. The first person selected is the chairperson; the second, the vice-chairperson; the third, the secretary; and the fourth, the treasurer. How many different leadership structures are possible? 116,280

48. Forming a Committee Four members from a 50-person committee are to be selected randomly to serve as chairperson, vice-chairperson, secretary, and treasurer. The first person selected is the chairperson; the second, the vice-chairperson; the third, the secretary; and the fourth, the treasurer. How many different leadership structures are possible? 50P_4

49. Lottery A lottery exists where balls numbered 1 to 25 are drawn. In how many ways can the top 9 horses finish the race? 303,600

50. Forming a Committee In the U.S. Senate, there are 21 members on the Committee on Banking, Housing, and Urban Affairs. Nine of these 21 members are selected to be on the Subcommittee on Economic Policy. How many different committee structures are possible for this subcommittee? 293,930

51. Simple Random Sample How many different simple random samples of size 5 can be obtained from a population whose size is 50? 2,118,760
52. **Simple Random Sample** How many different simple random samples of size 7 can be obtained from a population whose size is 100? \( \binom{100}{7} \)

53. **Children** A family has six children. If this family has exactly two boys, how many different birth and gender orders are possible? 15

54. **Children** A family has eight children. If this family has exactly three boys, how many different birth and gender orders are possible? \( \binom{8}{3} \)

55. **DNA Sequences** (See Example 12) How many distinguishable DNA sequences can be formed using three As, two Cs, two Gs, and three Ts? 25,200

56. **DNA Sequences** (See Example 12) How many distinguishable DNA sequences can be formed using one A, four Cs, three Gs, and four Ts? 135,600

57. **Landscape Design** A golf-course architect has four linden trees, five white birch trees, and two bald cypress trees to plant in a row along a fairway. In how many ways can the landscaper plant the trees in a row, assuming that the trees are evenly spaced? 6,890

58. **Starting Lineup** A baseball team consists of three outfielders, four infielders, a pitcher, and a catcher. Assuming that the outfielders and infielders are indistinguishable, how many batting orders are possible? 2,520

59. **Little Lotto** In the Illinois Lottery game Little Lotto, an urn contains balls numbered 1 to 39. From this urn, 5 balls are chosen randomly, without replacement. For a $1 bet, a player chooses one set of five numbers. To win, all five numbers must match those chosen from the urn. The order in which the balls are selected does not matter. What is the probability of winning Little Lotto with one ticket? \( \frac{1}{\binom{39}{5}} \)

60. **Mega Millions** In Mega Millions, an urn contains balls numbered 1 to 56, and a second urn contains balls numbered 1 to 46. From the first urn, 5 balls are chosen randomly, without replacement and without regard to order. From the second urn, 1 ball is chosen randomly. For a $1 bet, a player chooses one set of five numbers to match the balls selected from the first urn and one number to match the ball selected from the second urn. To win, all six numbers must match; that is, the player must match the first 5 balls selected from the first urn and the single ball selected from the second urn. What is the probability of winning the Mega Millions with a single ticket? \( \frac{1}{\binom{56}{5} \cdot \binom{46}{1}} \)

61. **Selecting a Jury** The grade appeal process at a university requires that a jury be structured by selecting five individuals randomly from a pool of eight students and ten faculty.
   (a) What is the probability of selecting a jury of all students? \( \frac{\binom{8}{5} \cdot \binom{10}{0}}{\binom{18}{5}} \)
   (b) What is the probability of selecting a jury of all faculty? \( \frac{\binom{8}{0} \cdot \binom{10}{5}}{\binom{18}{5}} \)
   (c) What is the probability of selecting a jury of two students and three faculty? \( \frac{\binom{8}{2} \cdot \binom{10}{3}}{\binom{18}{5}} \)

62. **Selecting a Committee** Suppose that there are 55 Democrats and 45 Republicans in the U.S. Senate. A committee of seven senators is to be formed by selecting members of the Senate randomly.
   (a) What is the probability that the committee is composed of all Democrats? \( \frac{55}{\binom{100}{7}} \)
   (b) What is the probability that the committee is composed of all Republicans? \( \frac{45}{\binom{100}{7}} \)
   (c) What is the probability that the committee is composed of three Democrats and four Republicans? \( \frac{\binom{55}{3} \cdot \binom{45}{4}}{\binom{100}{7}} \)

63. **Acceptance Sampling** Suppose that a shipment of 120 electronic components contains 4 defective components. To determine whether the shipment should be accepted, a quality-control engineer randomly selects 4 of the components and tests them. If 1 or more of the components is defective, the shipment is rejected. What is the probability that the shipment is rejected? \( \frac{1}{153} \)

64. **In the Dark** A box containing twelve 40-watt light bulbs and eighteen 60-watt light bulbs is stored in your basement. Unfortunately, the box is stored in the dark and you need two 60-watt bulbs. What is the probability of randomly selecting two 60-watt bulbs from the box? \( \frac{1}{153} \)

65. **Randomly Playing Songs** Suppose a compact disk (CD) you just purchased has 13 tracks. After listening to the CD, you decide that you like 5 of the songs. The random feature on your CD player will play each of the 13 songs once in a random order. Find the probability that among the first 4 songs played you like 2 of them; you like 3 of them; you like all 4 of them.
   (a) you like 2 of them; \( \frac{5 \cdot 8 \cdot 3 \cdot 2 \cdot 1}{13 \cdot 12 \cdot 11 \cdot 10} = 0.036 \)
   (b) you like 3 of them; \( \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{13 \cdot 12 \cdot 11 \cdot 10} = 0.011 \)
   (c) you like all 4 of them. \( \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{13 \cdot 12 \cdot 11 \cdot 10} = 0.007 \)

66. **Packaging Error** Through a manufacturing error, three cans marked “regular soda” were accidentally filled with diet soda and placed into a 12-pack. Suppose that three cans are randomly selected from the 12-pack.
   (a) Determine the probability that exactly two contain diet soda. \( \frac{3 \cdot 2 \cdot 10}{12 \cdot 11 \cdot 10} = 0.1227 \)
   (b) Determine the probability that exactly one contains diet soda. \( \frac{3 \cdot 11}{12 \cdot 11 \cdot 10} = 0.4909 \)
   (c) Determine the probability that all three contain diet soda. \( \frac{10}{12 \cdot 11 \cdot 10} = 0.0083 \)

67. **Three of a Kind** You are dealt 5 cards from a standard 52-card deck. Determine the probability of being dealt three of a kind (such as three aces or three kings) by answering the following questions:
   (a) How many ways can 5 cards be selected from a 52-card deck? \( \binom{52}{5} \)
   (b) Each deck contains 4 twos, 4 threes, and so on. How many ways can three of the same card be selected from the deck? \( \binom{4}{3} \cdot \binom{47}{2} \)
   (c) The remaining 2 cards must be different from the 3 chosen and different from each other. For example, if we drew three kings, the 4th card cannot be a king. After selecting the three of a kind, there are 12 different ranks of card remaining in the deck that can be chosen. If we have three kings, then we can choose twos, threes, and so on. Of the 12 ranks remaining, we choose 2 of them and then select one of the 4 cards in each of the two chosen ranks. How many ways can we select the remaining 2 cards? \( \binom{12}{2} \cdot \binom{4}{1} \cdot \binom{4}{1} \)
   (d) Use the General Multiplication Rule to compute the probability of obtaining three of a kind. That is, what is the probability of selecting three of a kind and two cards that are not like? \( \frac{\binom{52}{5} \cdot \binom{4}{3} \cdot \binom{47}{2} \cdot \binom{12}{2} \cdot \binom{4}{1} \cdot \binom{4}{1}}{\binom{52}{5} \cdot \binom{51}{2}} \)
68. **Two of a Kind** Follow the outline presented in Problem 67 to determine the probability of being dealt exactly one pair. \(0.4226\)

69. **Acceptance Sampling** Suppose that you have just received a shipment of 20 modems. Although you don’t know this, 3 of the modems are defective. To determine whether you will accept the shipment, you randomly select 4 modems and test them. If all 4 modems work, you accept the shipment. Otherwise, the shipment is rejected. What is the probability of accepting the shipment? \(0.4912\)

70. **Acceptance Sampling** Suppose that you have just received a shipment of 100 televisions. Although you don’t know this, 6 are defective. To determine whether you will accept the shipment, you randomly select 5 televisions and test them. If all 5 televisions work, you accept the shipment; otherwise, the shipment is rejected. What is the probability of accepting the shipment? \(0.7291\)

71. **Password Policy** According to the Sefton Council Password Policy (August 2007), the UK government recommends the use of “Environ passwords with the following format: consonant, vowel, consonant, consonant, vowel, consonant, number, number (for example, pinray45).”

(a) Assuming passwords are not case sensitive, how many such passwords are possible (assume that there are 5 vowels and 21 consonants)? \(486,202,500\)

(b) How many passwords are possible if they are case sensitive? \(420,434,796\)

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**TECHNOLOGY STEP-BY-STEP**

**Factorials, Permutations, and Combinations**

**TI-83/84 Plus**

**Factorials**

1. To compute \(7!\), type \(7\) on the HOME screen.
2. Press MATH, then highlight PRB, and then select 4:!. With \(7!\) on the HOME screen, press ENTER again.

**Permutations and Combinations**

1. To compute \(7P3\), type \(7\) on the HOME screen.
2. Press MATH, then highlight PRB, and then select 2:nPr.
3. Type 3 on the HOME screen, and press ENTER.

**Note:** To compute \(7C3\), select 3:nCr instead of \(2:nP3\).

**Excel**

**Factorials or Combinations**

1. Select Insert and then Function . . . .
2. Highlight Math & Trig in the Function category. For combinations, select COMBIN in the function name and fill in the appropriate cells. For a factorial, select FACT in the function name and fill in the appropriate cells.

**Permutations**

1. Select Insert and then Function . . . .
2. Highlight Statistical in the Function category. For permutations, select PERMUT in the function name and fill in the appropriate cells.

---

**5.6 PUTTING IT TOGETHER: WHICH METHOD DO I USE?**

**Objectives**

1. Determine the appropriate probability rule to use
2. Determine the appropriate counting technique to use

**1 Determine the Appropriate Probability Rule to Use**

Working with probabilities can be challenging. The number of different probability rules can sometimes seem daunting. The material presented in this chapter is meant to provide the basic building blocks of probability theory. Knowing when to use a particular rule takes practice. To aid you in this learning process, consider the flowchart provided in Figure 16 on the following page. While not all situations can be handled directly with the formulas provided, they can be combined and expanded to many more situations. A firm understanding of the basic probability rules will help in understanding much of the later material in the text.
What is the probability that the contestant picks a case worth at least $100,000?

**Approach:** We will follow the flowchart in Figure 16.

**Solution:** We are asked to find the probability of an event that is not compound. Therefore, we must decide among the empirical, classical, or subjective approaches for determining probability. The probability experiment in this situation is selecting a suitcase. Each prize amount is randomly assigned to one of the 26 suitcases, so the outcomes are equally likely. From the table we see that 7 of the cases contain at least $100,000. Letting $E =$ “worth at least $100,000,” we compute $P(E)$ using the classical approach.

$$P(E) = \frac{N(E)}{N(S)} = \frac{7}{26} = 0.269$$
**EXAMPLE 2**

**Probability: Which Rule Do I Use?**

**Problem:** According to a Harris poll in January 2008, 14% of adult Americans have one or more tattoos, 50% have pierced ears, and 65% of those with one or more tattoos also have pierced ears. What is the probability that a randomly selected adult American has one or more tattoos and pierced ears?

**Approach:** We will follow the flowchart in Figure 16.

**Solution:** We are asked to find the probability of a compound event involving ‘AND’. Letting $E$ “one or more tattoos” and $F$ “ears pierced,” we are asked to find $P(E \text{ and } F)$. We need to determine if the two simple events, $E$ and $F$, are independent. The problem statement tells us that $P(F) = 0.50$ and $P(F|E) = 0.65$. Because $P(F) \neq P(F|E)$, the two events are not independent. We can find $P(E \text{ and } F)$ using the General Multiplication Rule.

\[
P(E \text{ and } F) = P(E) \cdot P(F|E) = (0.14)(0.65) = 0.091
\]

So, the chance of selecting an adult American at random who has one or more tattoos and pierced ears is 9.1%.

---

2. **Determine the Appropriate Counting Technique to Use**

To determine the appropriate counting technique to use we need the ability to distinguish between a sequence of choices and arrangement of items. We also need to determine whether order matters in the arrangements. To help in deciding which counting rule to use, we provide the flowchart in Figure 17. Keep in mind that it may be necessary to use several counting rules in one problem.

**Figure 17**
EXAMPLE 4

Counting: Which Technique Do I Use?

Problem: On February 17, 2008, the Daytona International Speedway hosted the 50th running of the Daytona 500. Touted by many to be the most anticipated event in racing history, the race carried a record purse of almost $18.7 million. With 43 drivers in the race, in how many different ways could the top four finishers (1st, 2nd, 3rd, and 4th place) occur?

Approach: We will follow the flowchart in Figure 17.

Solution: We are asked to find the number of ways the top four finishers can be selected. We can view this as a sequence of choices, where the first choice is the first-place driver, the second choice is the second-place driver, and so on. There are 43 ways to pick the first driver, 42 ways to pick the second, 41 ways to pick the third, and 40 ways to pick the fourth. The number of choices at each stage is independent of previous choices, so we use the Multiplication Rule of Counting. The number of ways the top four finishers can occur is

\[ N(\text{top four}) = 43 \cdot 42 \cdot 41 \cdot 40 = 2,961,840 \]

We could also approach this problem as an arrangement of units. Since each race position is distinguishable, order matters in the arrangements. We are arranging the
43 drivers taken 4 at a time, so we are only considering a subset of \( r = 4 \) distinct drivers in each arrangement. Using our permutation formula, we get

\[
N(\text{top four}) = _{43}P_4 = \frac{43!}{(43 - 4)!} = \frac{43!}{39!} = 43 \times 42 \times 41 \times 40 = 2,961,840
\]

Again there are 2,961,840 different ways that the top four finishers could occur.

#### 5.6 ASSESS YOUR UNDERSTANDING

**Concepts and Vocabulary**

1. What is the difference between a permutation and a combination?
2. What method of assigning probabilities to a simple event uses relative frequencies? \( \text{Empirical} \)
3. Which type of compound event is generally associated with multiplication? Which is generally associated with addition? \('\text{AND}'; '\text{OR}'\)

**Skill Building**

4. Suppose that you roll a pair of dice 1,000 times and get seven 350 times. Based on these results, what is the probability that the next roll results in seven? 0.35

For Problems 5 and 6, let the sample space be \( S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \). Suppose that the outcomes are equally likely.

5. Compute the probability of the event \( E = \{1, 3, 5, 10\} \). 0.4
6. Compute the probability of the event \( F = \text{"a number divisible by three."} \) 0.3
7. List all permutations of five objects \( a, b, c, d, \) and \( e \) taken three at a time without replacement.

In Problems 8 and 9, find the probability of the indicated event if \( P(E) = 0.7 \) and \( P(F) = 0.2 \).

8. Find \( P(E \text{ or } F) \) if \( E \) and \( F \) are mutually exclusive. 0.9
9. Find \( P(E \text{ or } F) \) if \( P(E \text{ and } F) = 0.15 \). 0.75

In Problems 10–12, evaluate each expression.

10. \( \frac{6!2!}{4!} \) 60
11. \( _3P_3 \) 20
12. \( _4C_4 \) 126

13. Suppose that events \( E \) and \( F \) are independent, \( P(E) = 0.8 \) and \( P(F) = 0.5 \). What is \( P(E \text{ and } F) \)? 0.4
14. Suppose that \( E \) and \( F \) are two events and \( P(E \text{ and } F) = 0.4 \) and \( P(E) = 0.9 \). Find \( P(F|E) \).
15. Suppose that \( E \) and \( F \) are two events and \( P(E) = 0.9 \) and \( P(F|E) = 0.3 \). Find \( P(E \text{ and } F) \). 0.27
16. List all combinations of five objects \( a, b, c, d, \) and \( e \) taken three at a time without replacement.

**Applying the Concepts**

14. \( \frac{4}{9} = 0.444 \)

17. Soccer? In a survey of 500 randomly selected Americans, it was determined that 22 play soccer. What is the probability that a randomly selected American plays soccer? 0.044

18. Apartment Vacancy A real estate agent conducted a survey of 200 landlords and asked how long their apartments remained vacant before a tenant was found. The results of the survey are shown in the table. The data are based on information obtained from the U.S. Census Bureau.

<table>
<thead>
<tr>
<th>Duration of Vacancy</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>Less than 1 month</td>
<td>42</td>
</tr>
<tr>
<td>1 to 2 months</td>
<td>38</td>
</tr>
<tr>
<td>2 to 4 months</td>
<td>45</td>
</tr>
<tr>
<td>4 to 6 months</td>
<td>30</td>
</tr>
<tr>
<td>6 to 12 months</td>
<td>24</td>
</tr>
<tr>
<td>1 to 2 years</td>
<td>13</td>
</tr>
<tr>
<td>2 years or more</td>
<td>8</td>
</tr>
</tbody>
</table>

(a) Construct a probability model for duration of vacancy.
(b) Is it unusual for an apartment to remain vacant for 2 years or more? Yes
(c) Determine the probability that a randomly selected apartment is vacant for 1 to 4 months. 0.415
(d) Determine the probability that a randomly selected apartment is vacant for less than 2 years. 0.96

19. Seating Arrangements In how many ways can three men and three women be seated around a circular table (that seats six) assuming that women and men must alternate seats? 36

20. Starting Lineups Payton’s futsal team consists of 10 girls, but only 5 can be on the field at any given time (four fielders and a goalie).

(a) How many starting lineups are possible if either Payton or Jordyn must play goalie? 252
(b) How many starting lineups are possible if either Payton or Jordyn must play goalie? 252

21. Titanic Survivors The following data represent the survival data for the ill-fated Titanic voyage by gender. The males are adult males and the females are adult females.

<table>
<thead>
<tr>
<th></th>
<th>Male</th>
<th>Female</th>
<th>Child</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Survived</td>
<td>338</td>
<td>316</td>
<td>57</td>
<td>711</td>
</tr>
<tr>
<td>Died</td>
<td>1,352</td>
<td>109</td>
<td>52</td>
<td>1,513</td>
</tr>
<tr>
<td>Total</td>
<td>1,690</td>
<td>425</td>
<td>109</td>
<td>2,224</td>
</tr>
</tbody>
</table>

(a) If a passenger is selected at random, what is the probability that the passenger survived? 0.320
(b) If a passenger is selected at random, what is the probability that the passenger was female or a child? 0.369
(c) If a passenger is selected at random, what is the probability that the passenger was female or a child? 0.369
(d) If a passenger is selected at random, what is the probability that the passenger was female or survived? 0.369
(e) If a passenger is selected at random, what is the probability that the passenger was female or survived? 0.369
(f) If a female passenger is selected at random, what is the probability that she survived? 0.744
24. National Honor Society
The distribution of National Honor Society members among the students at a local high school is shown in the table. A student’s name is drawn at random.

<table>
<thead>
<tr>
<th>Class</th>
<th>Total</th>
<th>National Honor Society</th>
</tr>
</thead>
<tbody>
<tr>
<td>Senior</td>
<td>92</td>
<td>37</td>
</tr>
<tr>
<td>Junior</td>
<td>112</td>
<td>30</td>
</tr>
<tr>
<td>Sophomore</td>
<td>125</td>
<td>20</td>
</tr>
<tr>
<td>Freshman</td>
<td>120</td>
<td>0</td>
</tr>
</tbody>
</table>

(a) What is the probability that the student is a junior? 0.249
(b) What is the probability that the student is a senior, given that the student is in the National Honor Society? 0.425

25. Instant Winner
In 2002, Valerie Wilson won $1 million in a scratch-off game (Cool Million) from the New York lottery. Four years later, she won $1 million in another scratch-off game ($3,000,000 Jubilee), becoming the first person in New York state lottery history to win $1 million or more in a scratch-off game twice. In the first game, she beat odds of 1 in 5.2 million to win. In the second, she beat odds of 1 in 705,600.

(a) What is the probability that an individual would win $1 million in both games if they bought one scratch-off ticket from each game? \( \approx 0.00000000000027 \)
(b) What is the probability that an individual would win $1 million twice in the $3,000,000 Jubilee scratch-off game? \( \approx 0.0000000000000002 \)

26. Text Twist
In the game Text Twist, 6 letters are given and the player must form words of varying lengths using the letters provided. Suppose that the letters in a particular game are ENHSIC.

(a) How many different arrangements are possible using all 6 letters? 720
(b) How many different arrangements are possible using only 4 letters? 360
(c) The solution to this game has three 6-letter words. To advance to the next round, the player needs at least one of the 6-letter words. If the player simply guesses, what is the probability that he/she will get one of the 6-letter words on their first guess of six letters? 0.0042

27. Digital Music Players
According to an October 2004 survey, 51% of teens own an iPod or other MP3 player. If the probability that both a teen and his or her parent own an iPod or other MP3 player is 0.220, what is the probability that a parent owns such a device given that his or her teenager owns one? 0.431

28. Weather Forecast
The weather forecast says there is a 10% chance of rain on Thursday. Jim wakes up on Thursday and sees overcast skies. Since it has rained for the past three days, he believes that the chance of rain is more likely 60% or higher. What method of probability assignment did Jim use? Subjective

29. Essay Test
An essay test in European History has 12 questions. Students are required to answer 8 of the 12 questions. How many different sets of questions could be answered? 495

30. Exercise Routines
Todd is putting together an exercise routine and feels that the sequence of exercises can affect his overall performance. He has 12 exercises to select from, but only has enough time to do 9. How many different exercise routines could he put together? 79,833,600

31. New Cars
If the 2008 Hyundai Elantra has 2 transmission types, 3 vehicle styles, 2 option packages, 8 exterior color choices, and 2 interior color choices, how many different Elantras are possible? 129

32. Lingo
In the gameshow Lingo, the team that correctly guesses a mystery word gets a chance to pull two Lingo balls from a bin. Balls in the bin are labeled with numbers corresponding to the numbers remaining on their Lingo board. There are also three prize balls and three red “stopper” balls in the bin. If a stopper ball is drawn first, the team loses their second draw. To form a Lingo, the team needs five numbers in a vertical, horizontal, or diagonal row. Consider the sample Lingo board below for a team that has just guessed a mystery word.

<table>
<thead>
<tr>
<th>L</th>
<th>I</th>
<th>N</th>
<th>G</th>
<th>O</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>34</td>
<td>48</td>
<td>66</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>16</td>
<td>22</td>
<td>58</td>
<td>74</td>
</tr>
<tr>
<td>26</td>
<td>52</td>
<td>40</td>
<td>70</td>
<td></td>
</tr>
</tbody>
</table>

(a) What is the probability that the first ball selected is on the Lingo board? 0.714
(b) What is the probability that the team draws a stopper ball on its first draw? 0.143
(c) What is the probability that the team makes a Lingo on their first draw? 0.0042
(d) What is the probability that the team makes a Lingo on their second draw? 0.0143
Chapter 5 Review

Summary

In this chapter, we introduced the concept of probability. Probability is a measure of the likelihood of a random phenomenon or chance behavior. Because we are measuring a random phenomenon, there is short-term uncertainty. However, this short-term uncertainty gives rise to long-term predictability.

Probabilities are numbers between zero and one, inclusive. The closer a probability is to one, the more likely the event is to occur. If an event has probability zero, it is said to be impossible. Events with probability one are said to be certain.

We introduced three methods for computing probabilities: (1) the empirical method, (2) the classical method, and (3) subjective probabilities. Empirical probabilities rely on the relative frequency with which an event happens. Classical probabilities require that an experiment be performed, whereas classical probability does not. Subjective probabilities are probabilities based on the opinion of the individual providing the probability. They are educated guesses about the likelihood of an event occurring, but still represent a legitimate way of assigning probabilities.

Vocabulary

- Probability (p. 259)
- Outcome (p. 259)
- The Law of Large Numbers (p. 259)
- Experiment (p. 260)
- Sample space (p. 260)
- Event (p. 260)
- Probability model (p. 261)
- Impossible event (p. 261)
- Certainty (p. 261)
- Unusual event (p. 261)

Formulas

- **Empirical Probability**
  \[ P(E) \approx \frac{\text{frequency of } E}{\text{number of trials of experiment}} \]

- **Classical Probability**
  \[ P(E) = \frac{\text{number of ways that } E \text{ can occur}}{\text{number of possible outcomes}} = \frac{N(E)}{N(S)} \]

- **Addition Rule for Disjoint Events**
  \[ P(E \text{ or } F) = P(E) + P(F) \]

- **General Addition Rule**
  \[ P(E \text{ or } F) = P(E) + P(F) - P(E \text{ and } F) \]

- **Probabilities of Complements**
  \[ P(E^c) = 1 - P(E) \]

- **Multiplication Rule for Independent Events**
  \[ P(E \text{ and } F) = P(E) \cdot P(F) \]

- **Multiplication Rule for \( n \) Independent Events**
  \[ P(E \text{ and } F \text{ and } G \ldots) = P(E) \cdot P(F) \cdot P(G) \cdot \ldots \]

We are also interested in probabilities of multiple outcomes. For example, we might be interested in the probability that either event \( E \) or event \( F \) happens. The Addition Rule is used to compute the probability of \( E \) or \( F \); the Multiplication Rule is used to compute the probability that both \( E \) and \( F \) occur. Two events are mutually exclusive (or disjoint) if they do not have any outcomes in common. That is, mutually exclusive events cannot happen at the same time. Two events \( E \) and \( F \) are independent if knowing that one of the events occurs does not affect the probability of the other. The complement of an event \( E \), denoted \( E^c \), is all the outcomes in the sample space that are not in \( E \).

Finally, we introduced counting methods. The Multiplication Rule of Counting is used to count the number of sequences of events that can occur. Permutations are used to count the number of ways \( r \) items can be arranged from a set of \( n \) distinct items. Combinations are used to count the number of ways \( r \) items can be selected from a set of \( n \) distinct items without replacement and without regard to order. These counting techniques can be used to calculate probabilities using the classical method.
## Objectives

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<tr>
<th>Section</th>
<th>You should be able to . . .</th>
<th>Examples</th>
<th>Review Exercises</th>
</tr>
</thead>
<tbody>
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<td>1 Apply the rules of probabilities (p. 260)</td>
<td>2</td>
<td>1, 13(d), 15</td>
</tr>
<tr>
<td></td>
<td>2 Compute and interpret probabilities using the empirical method (p. 262)</td>
<td>3, 4, 7(a)</td>
<td>14(a), 15, 16(a) and (b), 17(a), 30</td>
</tr>
<tr>
<td></td>
<td>3 Compute and interpret probabilities using the classical method (p. 263)</td>
<td>5, 6, 7(b)</td>
<td>2–4, 13(a) and (d), 32(a) and (b)</td>
</tr>
<tr>
<td></td>
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<td>8</td>
<td>27</td>
</tr>
<tr>
<td></td>
<td>5 Recognize and interpret subjective probabilities (p. 268)</td>
<td></td>
<td>28</td>
</tr>
<tr>
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<td>1 Use the Addition Rule for disjoint events (p. 274)</td>
<td>1 and 2</td>
<td>3, 4, 7, 13(b) and (c), 32(g)</td>
</tr>
<tr>
<td></td>
<td>2 Use the General Addition Rule (p. 277)</td>
<td>3 and 4</td>
<td>6, 16(d)</td>
</tr>
<tr>
<td></td>
<td>3 Compute the probability of an event using the Complement Rule (p. 279)</td>
<td>5 and 6</td>
<td>5, 14(b), 17(b)</td>
</tr>
<tr>
<td>5.3</td>
<td>1 Identify independent events (p. 286)</td>
<td>1</td>
<td>9, 16(g), 32(f)</td>
</tr>
<tr>
<td></td>
<td>2 Use the Multiplication Rule for independent events (p. 287)</td>
<td>2 and 3</td>
<td>14(c) and (d), 17(c) and (c), 18, 19</td>
</tr>
<tr>
<td></td>
<td>3 Compute at-least probabilities (p. 289)</td>
<td>4</td>
<td>14(e), 17(d) and (f)</td>
</tr>
<tr>
<td>5.4</td>
<td>1 Compute conditional probabilities (p. 292)</td>
<td>1 through 3</td>
<td>11, 16(f), 32(c) and (d)</td>
</tr>
<tr>
<td></td>
<td>2 Compute probabilities using the General Multiplication Rule (p. 295)</td>
<td>4 through 6</td>
<td>10, 20, 29</td>
</tr>
<tr>
<td>5.5</td>
<td>1 Solve counting problems using the Multiplication Rule (p. 302)</td>
<td>1 through 4</td>
<td>21</td>
</tr>
<tr>
<td></td>
<td>2 Solve counting problems using permutations (p. 305)</td>
<td>5 through 7</td>
<td>10(e) and (f), 22</td>
</tr>
<tr>
<td></td>
<td>3 Solve counting problems using combinations (p. 307)</td>
<td>8 through 11</td>
<td>10(c) and (d), 24</td>
</tr>
<tr>
<td></td>
<td>4 Solve counting problems involving permutations with non-distinct items (p. 310)</td>
<td>12 and 13</td>
<td>23</td>
</tr>
<tr>
<td></td>
<td>5 Compute probabilities involving permutations and combinations (p. 311)</td>
<td>14 and 15</td>
<td>25, 26</td>
</tr>
<tr>
<td>5.6</td>
<td>1 Determine the appropriate probability rule to use (p. 315)</td>
<td>1, 2</td>
<td>13–20, 25, 26, 29(c) and (d), 30</td>
</tr>
<tr>
<td></td>
<td>2 Determine the appropriate counting technique to use (p. 317)</td>
<td>3, 4</td>
<td>21–25</td>
</tr>
</tbody>
</table>

## Review Exercises

1. (a) Which among the following numbers could be the probability of an event?  
   0, 1, 0.89, 0.01, 0.75, 0.41, 1.34  
   Suppose that the outcomes are equally likely.

1. (b) Which among the following numbers could be the probability of an event?  
   2, 1, 4, 6, 3, 7, 5

For Problems 2–5, let the sample space be \( S = \{ \text{red, green, blue, orange, yellow} \} \). Suppose that the outcomes are equally likely.

2. Compute the probability of the event \( E = \{ \text{yellow} \} \).  
   \( \frac{1}{5} \)

3. Compute the probability of the event \( F = \{ \text{green or orange} \} \).  
   \( \frac{2}{5} \)

4. Compute the probability of the event \( E = \{ \text{red or blue or yellow} \} \).  
   \( \frac{3}{5} \)

5. Suppose that \( E = \{ \text{yellow} \} \). Compute the probability of \( E^c \).  
   \( \frac{4}{5} \)

6. Suppose that \( P(E) = 0.76 \), \( P(F) = 0.45 \), and \( P(E \text{ or } F) = 0.32 \). What is \( P(E \text{ or } F) \)?  
   \( 0.99 \)

7. Suppose that \( P(E) = 0.36 \), \( P(F) = 0.12 \), and \( E \) and \( F \) are mutually exclusive. What is \( P(E \text{ or } F) \)?  
   \( 0.48 \)

8. Suppose that events \( E \) and \( F \) are independent. In addition, \( P(E) = 0.45 \) and \( P(F) = 0.2 \). What is \( P(E \text{ and } F) \)?  
   \( 0.09 \)

9. Suppose that \( P(E) = 0.8 \), \( P(F) = 0.5 \), and \( P(E \text{ and } F) = 0.24 \). Are events \( E \) and \( F \) independent? Why?  
   No

10. Suppose that \( P(E) = 0.59 \) and \( P(F|E) = 0.45 \). What is \( P(E \text{ and } F) \)?  
    \( 0.2655 \)

11. Suppose that \( P(E \text{ and } F) = 0.35 \) and \( P(F) = 0.7 \). What is \( P(E|F) \)?  
    \( 0.5 \)

12. Determine the value of each of the following:  
    (a) \( 7! \) 5040  
    (b) \( 0! \) 1  
    (c) \( \binom{5}{4} \) 126  
    (d) \( \binom{10}{4} \) 210  
    (e) \( 12P_2 \) 72  
    (f) \( 10P_4 \) 11880

13. **Roulette** In the game of roulette, a wheel consists of 38 slots, numbered 0, 00, 1, 2, . . . . , 36. (See the photo in Problem 35 from Section 5.1.) To play the game, a metal ball is spun around the wheel and allowed to fall into one of the numbered slots. The slots numbered 0 and 00 are green, the odd numbers are red, and the even numbers are black.

   (a) Determine the probability that the metal ball falls into a green slot. Interpret this probability.
   (b) Determine the probability that the metal ball falls into a green or a red slot. Interpret this probability.
   (c) Determine the probability that the metal ball falls into 00 or a red slot. Interpret this probability.
   (d) Determine the probability that the metal ball falls into the number 31 and a black slot simultaneously. What term is used to describe this event?  
   (e) Determine the probability that the metal ball falls into the number 19 and a black slot simultaneously. What term is used to describe this event?  
   (f) Determine the probability that the metal ball falls into at least one of the odd numbered slots. What term is used to describe this event?  
   (g) Determine the probability that the metal ball falls into at least one of the even numbered slots. What term is used to describe this event?  
   (h) Determine the probability that the metal ball falls into at least one of the numbered slots which are not 0 or 00. What term is used to describe this event?  
   (i) Determine the probability that the metal ball falls into at least one of the numbered slots which are not 0, 00, or 36. What term is used to describe this event?  
   (j) Determine the probability that the metal ball falls into at least one of the numbered slots which are not 0, 00, 31, or 36. What term is used to describe this event?
14. New Year’s Holiday  Between 6:00 p.m. December 30, 2005, and 5:59 a.m. January 3, 2006, there were 454 traffic fatalities in the United States. Of these, 193 were alcohol related.

(a) What is the probability that a randomly selected traffic fatality that happened between 6:00 p.m. December 30, 2005, and 5:59 a.m. January 3, 2006, was alcohol related? 0.425

(b) What is the probability that a randomly selected traffic fatality that happened between 6:00 p.m. December 30, 2005, and 5:59 a.m. January 3, 2006, was not alcohol related? 0.575

(c) What is the probability that two randomly selected traffic fatalities that happened between 6:00 p.m. December 30, 2005, and 5:59 a.m. January 3, 2006, were both alcohol related? 0.018

(d) What is the probability that neither of two randomly selected traffic fatalities that happened between 6:00 p.m. December 30, 2005, and 5:59 a.m. January 3, 2006, were alcohol related? 0.330

(e) What is the probability that of two randomly selected traffic fatalities that happened between 6:00 p.m. December 30, 2005, and 5:59 a.m. January 3, 2006, at least one was alcohol related? 0.670

15. Memphis Workers  The following data represent the distribution of class of workers in Memphis, Tennessee, in 2006.

<table>
<thead>
<tr>
<th>Class of Worker</th>
<th>Number of Workers</th>
</tr>
</thead>
<tbody>
<tr>
<td>Private wage and salary worker</td>
<td>219,852</td>
</tr>
<tr>
<td>Government worker</td>
<td>39,351</td>
</tr>
<tr>
<td>Self-employed worker</td>
<td>17,060</td>
</tr>
<tr>
<td>Unpaid family worker</td>
<td>498</td>
</tr>
</tbody>
</table>


(a) Construct a probability model for Memphis workers.
(b) Is it unusual for a Memphis worker to be an unpaid family worker? Yes
(c) Is it unusual for a Memphis worker to be self-employed? No

16. Gestation Period versus Weight  The following data represent the birth weights (in grams) of babies born in 2005, along with the period of gestation.

<table>
<thead>
<tr>
<th>Birth Weight (grams)</th>
<th>Period of Gestation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Preterm</td>
</tr>
<tr>
<td>Less than 1000</td>
<td>29,764</td>
</tr>
<tr>
<td>1000–1999</td>
<td>84,791</td>
</tr>
<tr>
<td>2000–2999</td>
<td>252,116</td>
</tr>
<tr>
<td>3000–3999</td>
<td>145,506</td>
</tr>
<tr>
<td>4000–4999</td>
<td>8,747</td>
</tr>
<tr>
<td>Over 5000</td>
<td>192</td>
</tr>
<tr>
<td>Total</td>
<td>521,116</td>
</tr>
</tbody>
</table>


(a) What is the probability that a randomly selected baby born in 2005 was postterm? 0.058
(b) What is the probability that a randomly selected baby born in 2005 weighed 3,000 to 3,999 grams? 0.656
(c) What is the probability that a randomly selected baby born in 2005 weighed 3,000 to 3,999 grams and was postterm? 0.042
(d) What is the probability that a randomly selected baby born in 2005 weighed 3,000 to 3,999 grams or was postterm? 0.672
(e) What is the probability that a randomly selected baby born in 2005 weighed less than 1,000 grams and was postterm? Is this event impossible? No
(f) What is the probability that a randomly selected baby born in 2005 weighed 3,000 to 3,999 grams, given the baby was postterm? 0.722
(g) Are the events “postterm baby” and “weighs 3,000 to 3,999 grams” independent? Why? No

17. Better Business Bureau  The Better Business Bureau reported that approximately 72.2% of consumer complaints in 2006 were settled.

(a) If a consumer complaint from 2006 is randomly selected, what is the probability that it was settled? 0.722
(b) What is the probability that it was not settled? 0.278
(c) If a random sample of five consumer complaints is selected, what is the probability that all five were settled? 0.000004
(d) If a random sample of five consumer complaints is selected, what is the probability that at least one was not settled? 0.9996
(e) If a random sample of five consumer complaints is selected, what is the probability that none was settled? 0.000001
(f) If a random sample of five consumer complaints is selected, what is the probability that at least one was settled? 0.9994

18. Pick 3  For the Illinois Lottery’s PICK 3 game, a player must match a sequence of three repeatable numbers, ranging from 0 to 9, in exact order (for example, 3–7–2). With a single ticket, what is the probability of matching the three winning numbers? 0.001

19. Pick 4  The Illinois Lottery’s PICK 4 game is similar to PICK 3, except a player must match a sequence of four repeatable numbers, ranging from 0 to 9, in exact order (for example, 5–8–5–1). With a single ticket, what is the probability of matching the four winning numbers? 0.000001

20. Drawing Cards  Suppose that you draw 3 cards without replacement from a standard 52-card deck. What is the probability that all 3 cards are aces? 0.0018

21. Forming License Plates  A license plate is designed so that the first two characters are letters and the last four characters are digits (0 through 9). How many different license plates can be formed assuming that letters and numbers can be used more than once? 6,760,000

22. Choosing a Seat  If four students enter a classroom that has 10 vacant seats, in how many ways can they be seated? 5040

23. Arranging Flags  How many different vertical arrangements are there of 10 flags if 4 are white, 3 are blue, 2 are green, and 1 is red? 12,600

24. Simple Random Sampling  How many different simple random samples of size 8 can be obtained from a population whose size is 55? 55C8

25. Arizona’s Pick 5  In one of Arizona’s lotteries, balls are numbered 1 to 35. Five balls are selected randomly, without replacement. The order in which the balls are selected does not matter. To win, your numbers must match the five selected. Determine your probability of winning Arizona’s Pick 5 with one ticket. 1/122,184,900

17. (c) 0.196  17. (e) 0.002  17. (f) 0.998
26. Packaging Error Because of a mistake in packaging, a case of 12 bottles of red wine contained 5 Merlot and 7 Cabernet, each without labels. All the bottles look alike and have an equal probability of being chosen. Three bottles are randomly selected. 26. (b) 0.3182
(a) What is the probability that all three are Merlot? 0.0455
(b) What is the probability that exactly two are Merlot?
(c) What is the probability that none is a Merlot? 0.1591

27. Simulation Use a graphing calculator or statistical software to simulate the playing of the game of roulette, using an integer distribution with numbers 1 through 38. Repeat the simulation 100 times. Let the number 37 represent 0 and the number 38 represent 00. Use the results of the simulation to answer the following questions.
(a) What is the probability that the ball lands in the slot marked 0?
(b) What is the probability that the ball lands in the slot marked 0 or in the one marked 00?

28. Explain what is meant by a subjective probability. List some examples of subjective probabilities.

29. Playing Five-Card Stud In the game of five-card stud, one card is dealt face down to each player and the remaining four cards are dealt face up. After two cards are dealt (one down and one up), the players bet. Players continue to bet after each additional card is dealt. Suppose three cards have been dealt to each of five players at the table. You currently have three clubs in your hand, so you will attempt to get a flush (all cards in the same suit). Of the cards dealt, there are two clubs showing in other player’s hands. (a) How many clubs are in a standard 52-card deck? 13
(b) How many cards remain in the deck or are not known by you? Of this amount, how many are clubs? 41; 8
(c) What is the probability that you get dealt a club on the next card? 8/41
(d) What is the probability that you get dealt two clubs in a row? 0.034
(e) Should you stay in the game?

30. Mark McGwire During the 1998 major league baseball season, Mark McGwire of the St. Louis Cardinals hit 70 home runs. Of the 70 home runs, 34 went to left field, 20 went to left center field, 13 went to center field, 3 went to right center field, and 0 went to right field. Source: Miklasz, B., et al. Celebrating 70: Mark McGwire’s Historic Season, Sporting News Publishing Co., 1998, p. 179.

(a) What is the probability that a randomly selected home run was hit to left field? Interpret this probability. 0.486
(b) What is the probability that a randomly selected home run was hit to right field? 0
(c) Was it impossible for Mark McGwire to hit a homer to right field? No

31. Lottery Luck In 1996, a New York couple won $2.5 million in the state lottery. Eleven years later, the couple won $5 million in the state lottery using the same set of numbers. The odds of winning the New York lottery twice are roughly 1 in 16 trillion, described by a lottery spokesperson as “galactically astronomical.” Although it is highly unlikely that an individual will win the lottery twice, it is not unlikely that someone will win a lottery twice. Explain why this is the case.

32. Coffee Sales The following data represent the number of cases of coffee or filters sold by four sales reps in a recent sales competition.

<table>
<thead>
<tr>
<th>Salesperson</th>
<th>Gourmet</th>
<th>Single Cup</th>
<th>Filters</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Connor</td>
<td>142</td>
<td>325</td>
<td>30</td>
<td>497</td>
</tr>
<tr>
<td>Paige</td>
<td>42</td>
<td>125</td>
<td>40</td>
<td>207</td>
</tr>
<tr>
<td>Bryce</td>
<td>9</td>
<td>100</td>
<td>10</td>
<td>119</td>
</tr>
<tr>
<td>Mallory</td>
<td>71</td>
<td>75</td>
<td>40</td>
<td>186</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>264</strong></td>
<td><strong>625</strong></td>
<td><strong>120</strong></td>
<td><strong>1,009</strong></td>
</tr>
</tbody>
</table>

(a) What is the probability that a randomly selected case was sold by Bryce? Is this unusual? 0.118; no
(b) What is the probability that a randomly selected case was Gourmet? 0.262
(c) What is the probability that a randomly selected Single-Cup case was sold by Mallory? 0.120
(d) What is the probability that a randomly selected Gourmet case was sold by Bryce? Is this unusual? 0.034; yes
(e) What can be concluded from the results of parts (a) and (d)?
(f) Are the events “Mallory” and “Filters” independent? Explain. No
(g) Are the events “Paige” and “Gourmet” mutually exclusive? Explain. No

**CHAPTER TEST**

1. Which among the following numbers could be the probability of an event? 0.23, 0.3, 0.5; 1.32 0.23, 0.3/4

For Problems 2–4, let the sample space by \( S = \{Chris, Adam, Elaine, Brian, Jason\} \). Suppose that the outcomes are equally likely.

2. Compute the probability of the event \( E = \{Jason\} \). 1/5
3. Compute the probability of the event \( E = \{Chris or Elaine\} \). 0.5
4. Suppose that \( E = \{Adam\} \). Compute the probability of \( E^c \).
5. Suppose that \( P(E) = 0.37 \) and \( P(F) = 0.22 \).
(a) Find \( P(E \text{ or } F) \) if \( E \text{ and } F \) are mutually exclusive. 0.59
(b) Find \( P(E \text{ and } F) \) if \( E \text{ and } F \) are independent. 0.0814
3. 2/5 4. 4/5

6. Suppose that \( P(E) = 0.15 \), \( P(F) = 0.45 \), and \( P(F|E) = 0.70 \).
(a) What is \( P(E \text{ and } F) \)? 0.105
(b) What is \( P(E \text{ or } F) \)? 0.495
(c) What is \( P(E|F) \)? 0.233
(d) Are \( E \text{ and } F \) independent? No

7. Determine the value of each of the following:
(a) \( 8! = 40,320 \)
(b) \( \binom{9}{2} = 36 \)
(c) \( \mu_P_e = 121,080,960 \)

8. Craps is a dice game in which two fair dice are cast. If the roller shoots a 7 or 11 on the first roll, he or she wins. If the roller shoots a 2, 3, or 12 on the first roll, he or she loses. (a) Compute the probability that the shooter wins on the first roll. Interpret this probability. 2/9
(b) Find if \( P(E) \text{ and } F \).
(c) Find if \( P(E | F) \).
(d) What is the probability that a randomly selected home run was hit to right field? 0
(e) Was it impossible for Mark McGwire to hit a homer to right field? No
The following probability model shows the distribution of the most-popular-selling Girl Scout Cookies®.

<table>
<thead>
<tr>
<th>Cookie Type</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>Thin Mints</td>
<td>0.25</td>
</tr>
<tr>
<td>Samoas®/Caramel deLites™</td>
<td>0.19</td>
</tr>
<tr>
<td>Peanut Butter Patties®/Tagalongs™</td>
<td>0.13</td>
</tr>
<tr>
<td>Peanut Butter Sandwich/Do-si-dos™</td>
<td>0.11</td>
</tr>
<tr>
<td>Shortbread/Trefoils</td>
<td>0.09</td>
</tr>
<tr>
<td>Other varieties</td>
<td>0.23</td>
</tr>
</tbody>
</table>

Source: www.girlscouts.org

(a) Verify that this is a probability model.
(b) If a girl scout is selling cookies to people who randomly enter a shopping mall, what is the probability that the next box sold will be Peanut Butter Patties®/Tagalongs™ or Peanut Butter Sandwich/Do-si-dos™?
(c) If a girl scout is selling cookies to people who randomly enter a shopping mall, what is the probability that the next box sold will be Thin Mints, Samoas®/Caramel deLites™, or Shortbread/Trefoils?
(d) What is the probability that the next box sold will not be Thin Mints?

The following data represent the medal tallies of the top eight countries at the 2006 Winter Olympics in Turin, Italy.

<table>
<thead>
<tr>
<th>Country</th>
<th>Gold</th>
<th>Silver</th>
<th>Bronze</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Germany</td>
<td>11</td>
<td>12</td>
<td>6</td>
<td>29</td>
</tr>
<tr>
<td>USA</td>
<td>9</td>
<td>9</td>
<td>7</td>
<td>25</td>
</tr>
<tr>
<td>Austria</td>
<td>9</td>
<td>7</td>
<td>7</td>
<td>23</td>
</tr>
<tr>
<td>Russia</td>
<td>8</td>
<td>6</td>
<td>8</td>
<td>22</td>
</tr>
<tr>
<td>Canada</td>
<td>7</td>
<td>10</td>
<td>7</td>
<td>24</td>
</tr>
<tr>
<td>Sweden</td>
<td>7</td>
<td>2</td>
<td>5</td>
<td>14</td>
</tr>
<tr>
<td>Korea</td>
<td>6</td>
<td>3</td>
<td>2</td>
<td>11</td>
</tr>
<tr>
<td>Switzerland</td>
<td>5</td>
<td>4</td>
<td>5</td>
<td>14</td>
</tr>
<tr>
<td>Total</td>
<td>62</td>
<td>53</td>
<td>47</td>
<td>162</td>
</tr>
</tbody>
</table>

(a) If a medal is randomly selected from the top eight countries, what is the probability that it is gold?
(b) If a medal is randomly selected from the top eight countries, what is the probability that it was won by Russia?
(c) If a medal is randomly selected from the top eight countries, what is the probability that it is gold and was won by Russia?
(d) If a medal is randomly selected from the top eight countries, what is the probability that it is gold or was won by Sweden?
(e) If a bronze medal is randomly selected from the top eight countries, what is the probability that it won by Sweden?
(f) If a medal that was won by Sweden is randomly selected, what is the probability that it is bronze?

During the 2007 season, the Boston Red Sox won 59.3% of their games. Assuming that the outcomes of the baseball games are independent and that the percentage of wins this season will be the same as in 2007, answer the following questions:
(a) What is the probability that the Red Sox will win two games in a row?
(b) What is the probability that the Red Sox will win seven games in a row?
(c) What is the probability that the Red Sox will lose at least one of their next seven games?

You just received a shipment of 10 DVD players. One DVD player is defective. You will accept the shipment if two randomly selected DVD players work. What is the probability that you will accept the shipment?

In the game of Jumble, the letters of a word are scrambled. The player must form the correct word. In a recent game in a local newspaper, the Jumble “word” was LINCEY. How many different arrangements are there of the letters in this “word”?

A local area network requires eight characters for a password. The first character must be a letter, but the remaining seven characters can be either a letter or a digit (0 through 9). Lower- and uppercase letters are considered the same. How many passwords are possible for the local area network?

In 1 second, a 3-GHz computer processor can generate about 736,156 seconds (7.9 days). How long would it take such a processor to generate all the passwords for the scheme in Problem 17?
Sports Probabilities

Have you ever watched a sporting event on television in which the announcer cites an obscure statistic? Where do these numbers come from? Well, pretend that you are the statistician for your favorite sports team. Your job is to compile strange probabilities regarding your favorite team and a competing team. For example, during the 2007 baseball season, the Boston Red Sox won 68% of their day games. As statisticians, we represent this as a conditional probability as follows: Suppose that Boston was playing the Seattle Mariners during the day and that Seattle won 52% of the games that it played during the day. From these statistics, we predict that Boston will win the game. Other ideas for conditional probabilities include home versus road games, right- vs left-handed pitcher, weather, and so on. For basketball, consider conditional probabilities such as the probability of winning if the team’s leading scorer scores fewer than 12 points.

Use the statistics and probabilities that you compile to make a prediction about which team will win. Write an article that presents your predictions along with the supporting numerical facts. Maybe the article could include such “keys to the game” as “Our crack statistician has found that our football team wins 80% of its games when it holds opposing teams to less than 10 points.” Repeat this exercise for at least five games. Following each game, determine whether the team you chose has won or lost. Compute your winning percentage for the games you predicted. Did you predict the winner in more than 50% of the games?

The late spring morning broke along the banks of a river. Robert Donkin, an elderly retiree with fishing pole in hand, slipped through the underbrush that lined the river’s banks. As he neared the shore, he saw a rather large canvas bag floating in the water, held by foliage that leaned over the river. The bag appeared to be stuffed, well-worn, and heavily stained. Upon closer inspection, Mr. Donkin observed what he believed to be hair floating through the bag’s opening. Marking the spot of his discovery, the fisherman fetched the authorities.

Preliminary investigation at the scene revealed a body in the bag. Unfortunately, it was impossible to identify the corpse’s sex or race immediately. Estimating age was also out of the question. Forensics was assigned the task of identifying the victim and estimating the cause and time of death. While waiting for the forensics analysis, you, as the detective in charge, have gathered the information shown pages 327–328 concerning victim–offender relationships from recent reports from the FBI.

Using the information contained in the tables, you are to develop a preliminary profile of the victim and offender by determining the likelihood that:

1. The offender is at least 18.
2. The offender is white.
3. The offender is male.
4. The victim is a white female.
5. The victim is either white or female.
6. The victim and the offender are from the same age category.
7. The victim and the offender are from different age categories.
8. The victim and the offender are of the same race.
9. The victim and the offender are of different races.
10. The victim and the offender are of the same sex.
11. The victim and the offender are of different sexes.
12. Without knowing the contents of the forensic team’s report, what is your best prediction of the age, race, and sex of the victim? Explain your reasoning.
13. What is your best prediction as to the age, race, and sex of the offender? Explain your reasoning.

Soon after you finished this analysis, the preliminary forensics report was delivered to your desk. Although no identification had been made, the autopsy suggested that the cause of death was blunt-force trauma and that the body had been in the water at least 2 weeks. By using a variety of techniques, it was also determined that the victim was a white female with blonde hair. She was estimated as being in her mid-thirties, showed no signs of having had children, and was wearing no jewelry.

Based on this new information, you develop a new offender profile by determining the likelihood that:

14. The offender is at least 18.
15. The offender is white.
16. The offender is male.
### Murder Victims by Race and Sex

<table>
<thead>
<tr>
<th>Race</th>
<th>Male</th>
<th>Female</th>
<th>Unknown</th>
</tr>
</thead>
<tbody>
<tr>
<td>White</td>
<td>5,067</td>
<td>1,883</td>
<td>6</td>
</tr>
<tr>
<td>Black</td>
<td>6,294</td>
<td>1,126</td>
<td>1</td>
</tr>
<tr>
<td>Other</td>
<td>301</td>
<td>105</td>
<td>0</td>
</tr>
<tr>
<td>Unknown</td>
<td>131</td>
<td>42</td>
<td>34</td>
</tr>
</tbody>
</table>

### Sex of Victim by Race of Offender

<table>
<thead>
<tr>
<th>Race of Offender</th>
<th>Male</th>
<th>Female</th>
<th>Unknown</th>
</tr>
</thead>
<tbody>
<tr>
<td>White</td>
<td>2,162</td>
<td>1,126</td>
<td>30</td>
</tr>
<tr>
<td>Black</td>
<td>208</td>
<td>700</td>
<td>54</td>
</tr>
<tr>
<td>Other</td>
<td>54</td>
<td>28</td>
<td>27</td>
</tr>
<tr>
<td>Unknown</td>
<td>30</td>
<td>27</td>
<td>4</td>
</tr>
</tbody>
</table>

### Victim–Offender Relationship by Race

<table>
<thead>
<tr>
<th>Race of Victim</th>
<th>White</th>
<th>Black</th>
<th>Other</th>
<th>Unknown</th>
</tr>
</thead>
<tbody>
<tr>
<td>White</td>
<td>3,026</td>
<td>573</td>
<td>53</td>
<td>57</td>
</tr>
<tr>
<td>Black</td>
<td>208</td>
<td>3,034</td>
<td>12</td>
<td>49</td>
</tr>
<tr>
<td>Other</td>
<td>54</td>
<td>28</td>
<td>97</td>
<td>7</td>
</tr>
<tr>
<td>Unknown</td>
<td>30</td>
<td>27</td>
<td>4</td>
<td>26</td>
</tr>
</tbody>
</table>

### Sex of Victim by Race of Offender

<table>
<thead>
<tr>
<th>Race of Offender</th>
<th>Male</th>
<th>Female</th>
<th>Unknown</th>
</tr>
</thead>
<tbody>
<tr>
<td>Male</td>
<td>2,162</td>
<td>1,126</td>
<td>30</td>
</tr>
<tr>
<td>Female</td>
<td>1,126</td>
<td>700</td>
<td>54</td>
</tr>
<tr>
<td>Unknown</td>
<td>54</td>
<td>28</td>
<td>27</td>
</tr>
</tbody>
</table>

### Victim–Offender Relationship by Sex

<table>
<thead>
<tr>
<th>Sex of Offender</th>
<th>Male</th>
<th>Female</th>
<th>Unknown</th>
</tr>
</thead>
<tbody>
<tr>
<td>Male</td>
<td>4,710</td>
<td>490</td>
<td>89</td>
</tr>
<tr>
<td>Female</td>
<td>1,735</td>
<td>150</td>
<td>24</td>
</tr>
<tr>
<td>Unknown</td>
<td>50</td>
<td>11</td>
<td>26</td>
</tr>
</tbody>
</table>

### Race of Victim by Sex of Offender

<table>
<thead>
<tr>
<th>Sex of Offender</th>
<th>Male</th>
<th>Female</th>
<th>Unknown</th>
</tr>
</thead>
<tbody>
<tr>
<td>Male</td>
<td>3,322</td>
<td>330</td>
<td>57</td>
</tr>
<tr>
<td>Female</td>
<td>2,964</td>
<td>290</td>
<td>49</td>
</tr>
<tr>
<td>Unknown</td>
<td>159</td>
<td>20</td>
<td>7</td>
</tr>
</tbody>
</table>

### Source

17. The victim and the offender are from the same age category.
18. The victim and the offender are from different age categories.
19. The victim and the offender are of the same race.
20. The victim and the offender are of different races.
21. The victim and the offender are of the same sex.
22. The victim and the offender are of different sexes.
23. What is your best prediction of the age, race, and sex of the offender? Explain your reasoning.
24. Did your answers to the offender questions change once you knew the age, race, and sex of the victim? Explain.

Suppose that 45% of murder victims were known to be related to or acquainted with the offender, that 15% were murdered by an unrelated stranger, and that for 40% of victims relationship to their killer is unknown. Based on all the information available, complete your offender profile for this case.